Eur. Phys. J. C 45, 721–743 (2006)
Digital Object Identifier (DOI) 10.1140/epjc/s2005-02437-0

THE EUROPEAN PHYSICAL JOURNAL C

A systematization for one-loop 4D Feynman integrals

O.A. Battistel[1], G. Dallabona[2,a]

[1] Dept. de Física, Universidade Federal de Santa Maria, Cx. Postal 5093, 97119-900, Santa Maria, RS, Brasil
[2] Universidade Estadual Paulista - Instituto de Física Teórica, rua Pamplona 145, 01405-900, São Paulo, SP, Brasil

Received: 1 June 2005 / Revised version: 22 September 2005 /
Published online: 21 December 2005 –

Abstract. We present a strategy for the systematization of manipulations and calculations involving divergent (or not) Feynman integrals, typical of the one-loop perturbative solutions of QFT, where the use of an explicit regularization is avoided. Two types of systematization are adopted. The divergent parts are put in terms of a small number of standard objects, and a set of structure functions for the finite parts is also defined. Some important properties of the finite structures, specially useful in the verification of relations among Green's functions, are identified. We show that, in fundamental (renormalizable) theories, all the finite parts of two-, three- and four-point functions can be written in terms of only three basic functions while the divergent parts require (only) five objects. The final results obtained within the proposed strategy can be easily converted into those corresponding to any specific regularization technique providing an unified point of view for the treatment of divergent Feynman integrals. Examples of physical amplitudes evaluation and their corresponding symmetry relations verification are presented as well as generalizations of our results for the treatment of Green's functions having an arbitrary number of points are considered.

1 Introduction

The framework of quantum field theory (QFT) has, undoubtedly, become the main tool for the phenomenological description of fundamental interacting particles. Within this formalism, in principle, it is possible to investigate the physical consequences of any set of symmetries for the dynamics of any set of interacting fields. We have at our disposal a clear prescription to construct the corresponding Lagrangian density which is simultaneously invariant under space-time, global and local gauge symmetries that are supposed to be relevant. After this, the associated equations of motion for all the fields can be derived from the variational Hamilton principle. The solution of the equations thus obtained would allow us to describe in a detailed way all the pertinent physical processes. Unfortunately the last step cannot be performed in practical situations and we need to have recourse to perturbative methods for the solutions. The predictions are stated to a previously chosen order in the perturbative parameter. A set of Green's functions, which allows the connection among the external particles characterizing a specific physical process, needs to be evaluated. Such Green's functions are constructed following the corresponding Feynman rules involving propagators, vertex operators and combinatorial factors. Given the small number of ingredients, the perturbative amplitudes possess very general and common mathematical aspects. Thus it is possible to expect that the necessary manipulations and calculations may admit some systematization, in order to simplify the handling of the results for the finite as well as for the divergent parts of Feynman integrals.

Given the fact that almost all the definite predictions of a QFT go through the evaluation of perturbative amplitudes some manipulations involving divergent Feynman integrals are always necessary, an eventual simplification of such procedures would be very useful. This is precisely the purpose of the present contribution. We propose a calculational strategy [1] involving two types of systematization. The first one refers to the manipulations of divergent integrals which are made without adopting an explicit regularization so that the performed steps are also useful for the reader who wants to use a specific regularization technique. The results which we will present can be easily converted to the ones corresponding to any regularization method providing the evaluation of a small number of standard divergent objects [2]. The second type of systematization refers to the finite parts. A set of functions which are sufficient to write the one-loop physical parts of the amplitudes is identified. Studying relations among such a class of functions we will show that it is possible to reduce all the amplitudes to a small number of mathematical structures: one for each number of points or, equivalently, for each number of internal loop propagators in the perturbative Green's functions. Clearly, this feature allows us to obtain numerical results in a very simplified way, as we will show.

For the finite parts, we will also identify properties relating the referred functions corresponding to different number of points, which substantially simplifies the verification of symmetry relations involving perturbative physical amplitudes. Besides, such a decomposition emphasizes in a very

[a] e-mail: Dalla@ift.unesp.br

clear way physical aspects relative to unitarity which are contained in the perturbative amplitudes as well as it allows for a systematization for the study of several kinematical limits of physical interest. Such studies can be performed without the contamination of mathematically undefined quantities, ambiguities or regularization dependence as is usually made in the context of traditional 4D regularization techniques. Within the context of the dimensional regularization (DR) method [3], many of these undesirable aspects can be avoided in a consistent way. Due to this reason, a series of valuable works has been and continues to be done in order to construct useful systematizations specially to one-loop Feynman integrals. In particular, many years ago 't Hooft and Veltman [4] have derived a set of useful formulas for scalar (one-loop) integrals. At the same time Passarino and Veltman [5] studied tensor integrals and proposed a way to reduce tensor integrals to scalar ones. Alternative approaches were derived in [6–14]. This subject gained new interest in the literature specially due to practical limitations of the Passarino–Veltman approach in the treatment of integrals in massless theories because of infrared divergences. Another motivation is the increasing complexity of multi-particle experiments made in colliders which requires a more efficient systematization of one-loop Feynman integrals with a large number of points (external lines). In this direction there are many recent works in the literature (see, for example [11–20] and references therein) where the authors are worried mainly with the trouble of massless propagators.

The present work is organized as follows. In Sect. 2 we define a set of one-loop Feynman integrals which will be considered in the discussions. The calculational strategy used to manipulate and calculate the divergent integrals is discussed, and the divergent standard objects are defined in Sect. 3. In order to systematize the finite parts of the integrals, in Sect. 4 we introduce a set of structures and present some of their relevant properties, which are useful for a simple organization of the results for perturbative physical amplitudes as well as for a systematic verification of symmetry relations involving them. In Sect. 5 we perform the calculations of Feynman integrals in terms of the introduced structures for the divergent and finite structures. In Sect. 6 we consider the explicit evaluation of perturbative physical amplitudes which symmetry relations are considered in Sects. 7 and 8. Generalizations of our results to an arbitrary number of points are presented in Sect. 9 and, finally, in Sect. 10 we present our final remarks.

2 Basic one-loop Feynman integrals

We start observing that all perturbative amplitudes after performing eventual Dirac traces and/or other operations related to internal symmetries reduce to a combination of Feynman integrals. Some of them can be divergent structures. For the fundamental theories only a relatively small number of such undefined objects needs to be treated. The most simple ones are those with the most severe degrees of divergence; the one-point function integrals, which we define as

$$(I_1; I_1^{\mu}) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(1; k^{\mu})}{P(k_1, m)}, \tag{1}$$

where $P(k_i, m) = (k + k_i)^2 - m^2$. A cubic degree of divergence can be identified in $I_{1\mu}(k_1)$ as well as a quadratic one in $I_1(k_1)$. The two-point functions, on the other hand, can be written as combinations of the integrals

$$(I_2; I_2^{\mu}; I_2^{\mu\nu}) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(1; k^{\mu}; k^{\mu}k^{\nu})}{P(k_1, m)\, P(k_2, m)}. \tag{2}$$

The highest degree of divergence, the quadratic one, appears in $I_{2\mu\nu}(k_1, k_2)$. For the calculations of three-point functions we need to evaluate the structures

$$\left(I_3; I_3^{\mu}; I_3^{\mu\nu}; I_3^{\mu\nu\lambda}\right) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\left(1; k^{\mu}; k^{\mu}k^{\nu}; k^{\mu}k^{\nu}k^{\lambda}\right)}{P(k_1, m)\, P(k_2, m)\, P(k_3, m)}. \tag{3}$$

Some of the above structures are finite and the degree of divergence is not higher than the linear one. An analogous definition can be given for the four-point function integrals

$$\left(I_4; I_4^{\mu}; I_4^{\mu\nu}; I_4^{\mu\nu\lambda}; I_4^{\mu\nu\alpha\beta}\right) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\left(1; k^{\mu}; k^{\mu}k^{\nu}; k^{\mu}k^{\nu}k^{\lambda}; k^{\mu}k^{\nu}k^{\alpha}k^{\beta}\right)}{P(k_1, m)\, P(k_2, m)\, P(k_3, m)\, P(k_4, m)}. \tag{4}$$

The set of structures above is less problematic as far as the divergences are concerned due to the fact that only the logarithmic divergence is involved. In all previously defined integrals k_1, k_2, k_3 and k_4 stand for the momenta of the loop internal propagators which carry the mass m. Such arbitrary internal momenta are related, in physical amplitudes, to the external ones through their differences. Given the divergent character of almost all structures defined above, for the explicit evaluation we have to specify some philosophy to deal with the mathematically undefined quantities involved. Usually the calculations become reliable only after adopting a regularization technique. After this, in the intermediary steps, we invariably assume some specific consequences for the results which are intrinsically associated to the properties attributed to the divergent integrals by the adopted regularization. In the final form thus obtained for the amplitudes in general, it is not possible to specify in a clear way which are the particular effects of the adopted regularization for the results or, in other words, to know precisely in what extension the expression is dependent on the used technique. In order to perform an analysis as general as possible of the properties of divergent amplitudes, including their symmetry relations and the question of ambiguities related to the arbitrariness involved in the routing of the loop internal lines momenta, we have to avoid, as much as possible, specific choices in intermediary steps in such a way that all the possibilities remain still contained in the final results. If this is possible, we can change the usual focus of analysis, which is the verification by testing the consistency of the adopted regularization technique, to the identification of properties such a technique must have in order to be consistent. Having this in mind, we describe in the next section the calculational strategy we will adopt.

3 The strategy to handle divergent Feynman integrals

Instead of specifying a regularization, we will adopt an alternative strategy [1] to perform all the calculations. To justify all the intermediate manipulation, we will assume the presence of a generic regulating distribution only in an implicit way. This can be schematically represented by

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} f(k) \to \int \frac{\mathrm{d}^4 k}{(2\pi)^4} f(k) \left\{ \lim_{\Lambda_i^2 \to \infty} G_{\Lambda_i}\left(k, \Lambda_i^2\right) \right\} = \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} f(k). \quad (5)$$

Here the Λ_i are parameters of the generic distribution $G(\Lambda_i^2, k)$ which in addition to the obvious finite character of the modified integral must have two other very general properties. It must be even in the integrating momentum k, due to Lorentz invariance maintenance, and a well-defined connection limit must also exist, i.e.,

$$\lim_{\Lambda_i^2 \to \infty} G_{\Lambda_i}\left(k^2, \Lambda_i^2\right) = 1.$$

The first property implies that all integrals having odd integrands vanish, while the second one guarantees, in particular, that the value of finite integrals in the amplitudes will not be modified. Note that these requirements are completely general and are in agreement with any reasonable 4D regularization. After these assumptions we can manipulate the integrand of the divergent integrals by using identities to generate a mathematical expression where all the divergences are contained in momenta independent structures. Due to the fact that in perturbative amplitudes we always have propagators, an adequate identity to achieve this goal is the following:

$$\frac{1}{(k+k_i)^2 - m^2} = \sum_{j=0}^{N} \frac{(-1)^j \left(k_i^2 + 2k_i \cdot k\right)^j}{(k^2 - m^2)^{j+1}} + \frac{(-1)^{N+1} \left(k_i^2 + 2k_i \cdot k\right)^{N+1}}{(k^2 - m^2)^{N+1} \left[(k+k_i)^2 - m^2\right]}, \quad (6)$$

where k_i is (in principle) an arbitrary momentum used in the routing of an internal line. The value for N in the above expression can be adequately chosen to avoid unnecessary algebraic difficulty. It can be taken as the minor value that leads the last term in the above expression to a finite integral. As a consequence, all the momentum dependent parts of the amplitudes can be integrated without restrictions due to the connection limit requirement. In the divergent structures this way obtained, on the other hand, no additional assumptions are taken, and (in the present discussion) they are written as a combination of five objects, namely

$$\square_{\alpha\beta\mu\nu} = \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{24 k_\mu k_\nu k_\alpha k_\beta}{(k^2 - m^2)^4} - \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{4 g_{\alpha\beta} k_\mu k_\nu}{(k^2 - m^2)^3} - \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{4 g_{\alpha\nu} k_\beta k_\mu}{(k^2 - m^2)^3} - \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{4 g_{\alpha\mu} k_\beta k_\nu}{(k^2 - m^2)^3}, \quad (7)$$

$$\Delta_{\mu\nu} = \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{4 k_\mu k_\nu}{(k^2 - m^2)^3} - \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{g_{\mu\nu}}{(k^2 - m^2)^2}, \quad (8)$$

$$\nabla_{\mu\nu} = \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{2 k_\nu k_\mu}{(k^2 - m^2)^2} - \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{g_{\mu\nu}}{(k^2 - m^2)}, \quad (9)$$

$$I_{\log}(m^2) = \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2}, \quad (10)$$

$$I_{\mathrm{quad}}(m^2) = \int_\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2)}. \quad (11)$$

It is important to emphasize that with this strategy it becomes possible to map the final expressions obtained by us into the corresponding results of other techniques, due to the fact that all the steps are perfectly valid within reasonable regularization prescriptions, including the DR technique. All we need is to evaluate the divergent structures obtained in the specific philosophy which we want to contact.

Another aspect of the procedure is the similarity with the R-operation [21]; in the BPHZ [22] method with the subtraction procedures made in the dispersion relation technique. The identity (6), which addresses such a similarity, does not play crucial role but is only used to put the integrals in the desired form; the total dependence on the internal momenta must be located in finite terms. For this purpose any mathematical identity or procedure which allows us to obtain the appropriate structure is adequate. Subtraction procedures or a Taylor expansion in internal lines momenta are certainly efficient tools.

In order to conclude this section it is interesting to call the attention to another and crucial point, which refers to the consistency of the regulator. The two properties assumed, even in loop momentum and connection limit, are very general and must be fulfilled by all reasonable regularization. The conditions to be imposed over a specific choice for the regularization will emerge, in our procedure, in constraints stated to the basic divergent objects $\nabla_{\mu\nu}, \square_{\alpha\beta\mu\nu}$, and $\Delta_{\mu\beta}$ defined above, as will become clear in future sections. This aspect is extensively discussed in [2].

4 Basic structure functions for finite parts

As a consequence of application of the procedure described in the preceding section, we will invariably get finite integrals. After solving them, a careful analysis reveals that it is always possible to identify a set of basic functions, one for each number of points of the corresponding Green's functions. Such basic functions carry important information about general QFT aspects in perturbative solutions, like, for example, unitarity. Besides, due to the Ward identities, there must be relations among such basic functions corresponding to different number of points of the associated Green's functions. In addition, once we know that it is always possible to reduce all the tensor Feynman integrals, for each number of points, to only one scalar integral [5] it is expected to find the corresponding reduction at the level of the basic finite structure functions. Such a systematization could be very useful in the perturbative evaluation of the one-loop amplitudes in all theories and models just because in this type of reduction no handling of divergences is involved. In what follows we define such a set of structure functions and present the properties which allow a very simple systematization of the results for the Feynman integrals defined in (1)–(4) which we present in Sect. 5.

4.1 Basic two-point structure functions

The Feynman integrals having two denominators, defined in (2), after the use of the procedure described in Sect. 3, present finite integrals. In order to write the results in as simple as possible a way, it is convenient to introduce a set of basic functions which we define as

$$Z_k(p^2, m_1^2, m_2^2; \lambda^2) = \int_0^1 \mathrm{d}x \left[x^k\right] \ln\left(\frac{Q(p,x)}{-\lambda^2}\right), \qquad (12)$$

where $Q(p,x) = p^2x(1-x)+(m_1^2-m_2^2)x-m_1^2$. In the above definition, p is a momentum carried by an internal line or a combination of them, m_1 and m_2 are masses carried by the propagators and λ is a parameter with the dimension of mass, which plays the role of a common scale for all the involved physical quantities. In order to simplify the notation, from now on we will adopt $Z_0\left(p^2, m^2, m^2; m^2\right) = Z_0(p^2;m^2)$ once we are dealing with only one species of intermediate fermion.

The integration on the Feynman parameter x can be easily performed. Proceeding this way we can write the results as

$$Z_0\left(p^2;m^2\right) = -2 - \frac{h\left(p^2;m^2\right)}{2p^2}, \qquad (13)$$

$$Z_1\left(p^2;m^2\right) = -1 - \frac{h\left(p^2;m^2\right)}{4p^2}, \qquad (14)$$

$$Z_2\left(p^2;m^2\right) = -\frac{1}{18} - \frac{2\left(p^2-m^2\right)}{3p^2} - \frac{\left(p^2-m^2\right)}{6p^4} h\left(p^2;m^2\right), \qquad (15)$$

where $h(p^2;m^2)$ possesses three representations:
(i) $p^2 < 0$:

$$h\left(p^2;m^2\right) = 2\sqrt{-p^2}\sqrt{4m^2-p^2}\ln\left\{\frac{\sqrt{4m^2-p^2}-\sqrt{-p^2}}{\sqrt{-p^2}+\sqrt{-p^2}}\right\}, \qquad (16)$$

(ii) $0 < p^2 < 4m^2$:

$$h\left(p^2;m^2\right) = -4\sqrt{p^2}\sqrt{4m^2-p^2}\arctan\left\{\frac{\sqrt{p^2}}{\sqrt{4m^2-p^2}}\right\}, \qquad (17)$$

(iii) $p^2 > 4m^2$:

$$h\left(p^2;m^2\right) = 2\sqrt{p^2}\sqrt{p^2-4m^2}\ln\left\{\frac{\sqrt{p^2}-\sqrt{p^2-4m^2}}{\sqrt{p^2}+\sqrt{p^2-4m^2}}\right\} + 2\mathrm{i}\pi\sqrt{p^2}\sqrt{p^2-4m^2}, \qquad (18)$$

where we see that the function $h(p^2;m^2)$ develops an imaginary part as is required by the unitarity. Given the expressions above, it is immediate to identify the relations among the $Z_k(p^2;m^2)$ functions having different values for the index k. The first of such relations are

$$Z_1\left(p^2;m^2\right) = \frac{1}{2}Z_0\left(p^2;m^2\right), \qquad (19)$$

$$Z_2\left(p^2;m^2\right) = -\frac{1}{18} + \frac{2}{3}Z_1\left(p^2;m^2\right) - \frac{m^2}{3p^2}Z_0\left(p^2;m^2\right), \qquad (20)$$

$$Z_3\left(p^2;m^2\right) = -\frac{1}{24} + \frac{3}{4}Z_2\left(p^2;m^2\right) - \frac{m^2}{2p^2}Z_1\left(p^2;m^2\right), \qquad (21)$$

which show that for two-point functions all the results can be reduced to only $Z_0(p^2;m^2)$. Such kinds of relations are specially useful in verifications of Ward identities involving two-point functions.

4.2 Basic three-point structure functions

When the evaluation of the finite parts of the Feynman integrals defined in (3) is in order, it is interesting to introduce the following functions:

$$\xi_{nm}(p,q) = \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \frac{x_1^n x_2^m}{Q(p,x_1;q,x_2)}, \qquad (22)$$

where p and q are momenta of internal lines or a combination of them, and

$$Q(p,x_1;q,x_2) = p^2x_1(1-x_1) + q^2x_2(1-x_2) - 2(p\cdot q)x_1x_2 - m^2.$$

Before considering some important properties of the above defined functions, it is also interesting to introduce another set of functions related to that (see Sect. 9). They are defined as

$$\eta_{nm}\left(p,q;m^2\right) \qquad (23)$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \left[x_1^n x_2^m\right] \ln\left(\frac{Q(p,x_1;q,x_2)}{-m^2}\right).$$

The first important aspect related to the functions $\xi_{nm}(p,q)$ ($\eta_{nm}(p,q;m^2)$) is the reduction of all of them only in terms of $\xi_{00}(p,q)$ ($\eta_{00}(p,q;m^2)$). Let us start with those for which $n+m=1$. The first of them may be $\xi_{01}(p,q)$. After some algebraic effort, which involves basically integration by parts, we can write, for the case of equal masses,

$$\begin{aligned}\xi_{01}(p,q) &= \frac{C_1}{2}\left\{\left(\frac{p\cdot q-p^2}{p^2q^2}\right)Z_0\left((p-q)^2;m^2\right)\right.\\ &\quad -\frac{p\cdot q}{p^2q^2}Z_0\left(q^2;m^2\right) \\ &\quad \left.+\frac{1}{q^2}Z_0\left(p^2;m^2\right)+\left(\frac{q^2-p\cdot q}{q^2}\right)\xi_{00}(p,q)\right\}.\end{aligned} \tag{24}$$

Here we define $C_1 = p^2q^2\left[p^2q^2-(p\cdot q)^2\right]^{-1}$. On the other hand, $\xi_{10}(p,q)$ can be written as

$$\begin{aligned}\xi_{10}(p,q) &= \frac{C_1}{2}\left\{\left(\frac{p\cdot q-q^2}{p^2q^2}\right)Z_0\left((p-q)^2;m^2\right)\right.\\ &\quad -\frac{p\cdot q}{p^2q^2}Z_0\left(p^2;m^2\right)+\frac{1}{p^2}Z_0\left(q^2;m^2\right)\\ &\quad \left.+\left(\frac{p^2-p\cdot q}{p^2}\right)\xi_{00}(p,q)\right\}.\end{aligned} \tag{25}$$

In the last two equations above, we can note that both functions are related. In fact there is a general property which relates $\xi_{nm}(p,q)$ to $\xi_{mn}(p,q)$ which is the interchanging symmetry $p \leftrightarrow q$.

In order to give the corresponding explicit expressions for the $\xi_{nm}(p,q)$ functions corresponding to $n+m=2$ it is interesting first to develop the $\eta_{00}(p,q;m^2)$. Such a function can be written as

$$\begin{aligned}\eta_{00}(p,q;m^2) &= \frac{1}{2}Z_0\left((p-q)^2;m^2\right)-\left[\frac{1}{2}+m^2\xi_{00}(p,q)\right]\\ &\quad +\frac{1}{2}p^2\xi_{10}(p,q)+\frac{1}{2}q^2\xi_{01}(p,q).\end{aligned} \tag{26}$$

Now the explicit form for the function $\xi_{20}(p,q)$ and $\xi_{02}(p,q)$ may be given by

$$\begin{aligned}\xi_{02}(p,q) &= \frac{C_1}{2}\left\{\frac{(p\cdot q)}{2p^2q^2}\left[Z_0\left((p-q)^2;m^2\right)-Z_0\left(q^2;m^2\right)\right]\right.\\ &\quad -\frac{1}{q^2}\left[\frac{1}{2}Z_0\left((p-q)^2;m^2\right)-\eta_{00}\left(p,q;m^2\right)\right]\\ &\quad \left.+\left(\frac{q^2-p\cdot q}{q^2}\right)\xi_{01}(p,q)\right\},\end{aligned} \tag{27}$$

and $\xi_{20}(p,q) = \xi_{02}(q,p)$. On the other hand, we note that $\xi_{11}(p,q)$ admits two alternative forms. The first is

$$\begin{aligned}\xi_{11}(p,q) &= \frac{C_1}{2}\left\{\frac{-1}{2p^2}\left[Z_0\left((p-q)^2;m^2\right)-Z_0\left(q^2;m^2\right)\right]\right.\\ &\quad +\frac{(p\cdot q)}{p^2q^2}\left[\frac{1}{2}Z_0\left((p-q)^2;m^2\right)-\eta_{00}(p,q)\right]\\ &\quad \left.+\left(\frac{p^2-p\cdot q}{p^2}\right)\xi_{01}(p,q)\right\},\end{aligned} \tag{28}$$

while the second one is obtained by interchanging $p \leftrightarrow q$. Next we can give explicit expressions for the $\xi_{nm}(p,q)$ functions for $n+m=3$. For this purpose it is convenient first to develop the $\eta_{nm}(p,q;m^2)$ functions for $n+m=1$. We get

$$\begin{aligned}\eta_{10}(p,q;m^2) &= \frac{2}{3}\left\{\frac{1}{4}Z_0\left((p-q)^2;m^2\right)-\left[\frac{1}{6}+m^2\xi_{10}(p,q)\right]\right.\\ &\quad \left.+\frac{1}{2}q^2\xi_{11}(p,q)+\frac{1}{2}p^2\xi_{20}(p,q)\right\},\end{aligned} \tag{29}$$

and $\eta_{10}(p,q;m^2) = \eta_{01}(q,p;m^2)$. Now we first write $\xi_{30}(p,q)$ as

$$\begin{aligned}\xi_{30}(p,q) &= \frac{C_1}{2}\left\{\frac{(p\cdot q)}{p^2q^2}\left[Z_2\left((p-q)^2;m^2\right)-Z_2\left(p^2;m^2\right)\right]\right.\\ &\quad -\frac{1}{p^2}\left[Z_2\left((p-q)^2;m^2\right)-2\eta_{10}\left(p,q;m^2\right)\right]\\ &\quad \left.+\left(\frac{p^2-(p\cdot q)}{p^2}\right)\xi_{20}(p,q)\right\},\end{aligned} \tag{30}$$

and $\xi_{03}(q,p) = \xi_{30}(p,q)$. On the other hand, $\xi_{21}(p,q)$ can be given by

$$\begin{aligned}\xi_{21}(p,q) &= \frac{C_1}{2}\left\{-\frac{1}{q^2}\left[Z_2\left((p-q)^2;m^2\right)-Z_2\left(p^2;m^2\right)\right]\right.\\ &\quad +\frac{(p\cdot q)}{p^2q^2}\left[Z_2\left((p-q)^2;m^2\right)-2\eta_{10}\left(p,q;m^2\right)\right]\\ &\quad \left.+\left(\frac{q^2-(p\cdot q)}{q^2}\right)\xi_{20}(p,q)\right\},\end{aligned} \tag{31}$$

and $\xi_{12}(q,p) = \xi_{21}(p,q)$. At this point it is interesting to note that the functions $\xi_{nm}(p,q)$ corresponding to a certain value for $n+m$ are written in terms of those having the summation $n+m$ decreased by one unity, which are then given in terms of those corresponding to $n+m$ decreased by two units and so on until in the end, all of them are in fact combinations of $Z_k(p^2;m^2)$ functions (which can be reduced to $Z_0(p^2;m^2)$) plus $\xi_{00}(p,q)$. In special kinematical situations the function $\xi_{00}(p,q)$ can be decomposed in a summation involving $Z_k(p^2;m^2)$ having negative values for k. A simple illustration can be given in the situation $p^2=q^2=0$ (external massless particles on the mass shell). In this case we get

$$\xi_{00}=\frac{Z_{-1}(S;m^2)}{S},$$

where $S=-2(p\cdot q)$. However, in general $\xi_{00}(p,q)$ can be written in terms of $Z_{-k}(p^2;m^2)$ plus a term which cannot be decomposed into two-point functions structures in all the kinematical situations. For the purposes of the present work we will not need such kind of relations. At this point it is interesting to observe, given the preceding comments, that the functions $\xi_{nm}(p,q)$ possess imaginary parts with thresholds at the kinematical points (here m_1, m_2 and m_3 are masses carried by the propagators),

$$p^2=(m_1+m_2)^2, \tag{32}$$

$$q^2=(m_1+m_3)^2, \tag{33}$$

$$(p-q)^2=(m_2+m_3)^2, \tag{34}$$

which can be easily noted in the decompositions of $\xi_{nm}(p,q)$ corresponding to a specific value of $n+m$ in terms of those having $n+m-1$, where the functions $Z_k(p^2;m^2)$ appear with their thresholds. Such properties, as it is well-known, are required by unitarity: the amplitudes must develop an imaginary part at the kinematical point where both particles at a vertex are on their mass shell. It is also important to note that the $\xi_{nm}(p,q)$ functions are not defined at the kinematical points where one of the momenta is taken to zero or at the kinematical situation where both momenta are equal. This fact can be easily noted if we observe that, in these situations the functions $\xi_{nm}(p,q)$ become derivative of $Z_k(p^2;m^2)$ which is not defined at the complex threshold. Such situations are related to soft limits for the external particles. After these important remarks we now consider some properties of the $\xi_{nm}(p,q)$ functions which are very useful when Ward identities verifications are in order. For these purposes it is interesting to note that some combinations of the explicit expressions given for $\xi_{nm}(p,q)$ functions are reduced to simple expressions. They are

(i) $n+m=1$:

$$q^2\xi_{01}(p,q)+(p\cdot q)\,\xi_{10}(p,q) =-\frac{1}{2}Z_0\left((p-q)^2;m^2\right) +\frac{1}{2}Z_0\left(p^2;m^2\right)+\frac{1}{2}q^2\xi_{00}(p,q), \tag{35}$$

(ii) $n+m=2$:

$$q^2\xi_{02}(p,q)+(p\cdot q)\,\xi_{11}(p,q) =-\frac{1}{4}Z_0\left((p-q)^2;m^2\right) +\frac{1}{2}\eta_{00}\left(p,q;m^2\right)+\frac{1}{2}q^2\xi_{01}(p,q), \tag{36}$$

$$q^2\xi_{11}(p,q)+(p\cdot q)\,\xi_{20}(p,q) =-\frac{1}{2}Z_1\left((p-q)^2;m^2\right) +\frac{1}{2}Z_1\left(p^2;m^2\right)+\frac{1}{2}q^2\xi_{10}(p,q), \tag{37}$$

(iii) $n+m=3$:

$$q^2\xi_{21}(p,q)+(p\cdot q)\,\xi_{30}(p,q) =-\frac{1}{2}Z_2\left((p-q)^2;m^2\right) +\frac{1}{2}Z_2\left(p^2;m^2\right)+\frac{1}{2}q^2\xi_{20}(p,q), \tag{38}$$

$$q^2\xi_{03}(p,q)+(p\cdot q)\,\xi_{12}(p,q) =-\frac{1}{2}Z_2\left((p-q)^2;m^2\right) +\eta_{01}\left(p,q;m^2\right)+\frac{1}{2}q^2\xi_{02}(p,q), \tag{39}$$

$$q^2\xi_{12}(p,q)+(p\cdot q)\,\xi_{21}(p,q) =\frac{1}{2}Z_2\left((p-q)^2;m^2\right)-\frac{1}{4}Z_0\left((p-q)^2;m^2\right) +\frac{1}{2}\eta_{10}\left(p,q;m^2\right)+\frac{1}{2}q^2\xi_{11}(p,q). \tag{40}$$

In addition, it is also useful to note similar relations involving the $\eta_{nm}(p,q;m^2)$ functions like, for example,

$$q^2\eta_{01}\left(p,q;m^2\right)+(p\cdot q)\,\eta_{10}\left(p,q;m^2\right) =-(p-q)^2 \times\left[Z_2\left((p-q)^2;m^2\right)-\frac{1}{4}Z_0\left((p-q)^2;m^2\right)\right] +p^2\left[Z_2\left(p^2;m^2\right)-\frac{1}{4}Z_0\left(p^2;m^2\right)\right] +\frac{1}{2}q^2\eta_{00}\left(p,q;m^2\right). \tag{41}$$

Another set of relations can be obtained by changing $p\leftrightarrow q$ in the relations above and using the properties of the functions $\xi_{nm}(p,q)$ and $\eta_{nm}(p,q;m^2)$ under this symmetry. As an example consider the relation (35) with the change $p\leftrightarrow q$. In this case, since $\xi_{01}(p,q)\leftrightarrow\xi_{10}(p,q)$, ξ_{00} and Z_0 remain unchanged, we get

$$p^2\xi_{10}(p,q)+(p\cdot q)\,\xi_{01}(p,q) =-\frac{1}{2}Z_0\left((p-q)^2;m^2\right)+\frac{1}{2}Z_0\left(q^2;m^2\right) +\frac{1}{2}p^2\xi_{00}(p,q). \tag{42}$$

Furthermore, note that when on the left hand side we have $\xi_{nm}(p,q)$ having $n+m=3$, on the right hand side we will have only functions with $n+m=2$, and so on. Such type of structures are precisely the expected ones when the Ward identities are considered. It is clear that other functions corresponding to higher values of n and m, and analogous relations among them can be obtained. In the final section, Sect. 9, we will show how to generalize all the above functions and their relations to an arbitrary number of points. For the present purposes the $\xi_{nm}(p,q)$ given above will be enough.

4.3 Basic four-point structure functions

In the same way that the explicit calculation of three-point functions admits a systematization in terms of $\xi_{nm}(p,q)$ and $\eta_{nm}(p,q;m^2)$ functions, it is possible to introduce an analogous set of basic functions. They are defined as

$$\zeta_{nml}(p,q,r) = \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int_0^{1-x_1-x_2} \mathrm{d}x_3 \times \frac{x_1^n x_2^m x_3^l}{\left[Q(p,x_1;q,x_2;r,x_3)\right]^2}, \tag{43}$$

$$\xi_{nml}(p,q,r) = \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int_0^{1-x_1-x_2} \mathrm{d}x_3 \frac{x_1^n x_2^m x_3^l}{Q(p,x_1;q,x_2;r,x_3)}, \tag{44}$$

$$\eta_{nml}(p,q,r;m^2) = \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int_0^{1-x_1-x_2} \mathrm{d}x_3 \times \left[x_1^n x_2^m x_3^l\right] \ln\left(\frac{Q(p,x_1;q,x_2;r,x_3)}{-m^2}\right), \tag{45}$$

where

$$Q(p,x_1;q,x_2;r,x_3) = p^2 x_1(1-x_1) + q^2 x_2(1-x_2) + r^2 x_3(1-x_3) -2(p\cdot q)x_1x_2 - 2(p\cdot r)x_1x_3 - 2(q\cdot r)x_2x_3 -m^2.$$

All the functions of the set $\zeta_{nml}(p,q,r)$ can be, in the end, reduced to the most simple ones, $\zeta_{000}(p,q,r)$ and (more ξ_{00} and Z_0 functions). The functions $\xi_{nml}(p,q,r)$, on the other hand, can be reduced to $\xi_{000}(p,q,r)$ (and η_{00}). As examples of such reductions let us consider those corresponding to $n+m+l=1$. They can be written as follows.

(i) Functions ζ_{nml}:

$$\begin{aligned}\zeta_{100}(p,q,r) &= C_2^{-1}\Big\{\left[q^2r^2-(q\cdot r)^2\right]\left[\xi_{00}(s,u)-\xi_{00}(q,r)\right] \\ &+\left[(p\cdot r)(q\cdot r)-r^2(p\cdot q)\right] \\ &\times\left[\xi_{00}(s,-t)-\xi_{00}(p,r)\right] \\ &+\left[(p\cdot q)(q\cdot r)-q^2(p\cdot r)\right]\left[\xi_{00}(u,t)-\xi_{00}(p,q)\right] \\ &+\Big[p^2\left(q^2r^2-(q\cdot r)^2\right) \\ &+q^2\left((p\cdot r)(q\cdot r)-r^2(p\cdot q)\right) \\ &+r^2\left((p\cdot q)(q\cdot r)-q^2(p\cdot r)\right)\Big]\zeta_{000}(p,q,r)\Big\}.\end{aligned} \tag{46}$$

Note that $\zeta_{010}(p,q,r)=\zeta_{100}(q,p,r)$ and $\zeta_{001}(p,q,r)=\zeta_{100}(r,q,p)$. In the above expression we have defined

$$C_2 = 2\Big\{p^2q^2r^2+2(p\cdot q)(p\cdot r)(q\cdot r) -p^2(q\cdot r)^2-q^2(p\cdot r)^2-r^2(p\cdot q)^2\Big\},$$

and $s=p-q$, $u=p-r$ and $t=q-r$.

(ii) Functions ξ_{nml}:

$$\begin{aligned}\xi_{100}(p,q,r) &= C_2^{-1}\Big\{\left[q^2r^2-(q\cdot r)^2\right] \\ &\times\left[\eta_{00}(q,r;m^2)-\eta_{00}(s,u;m^2)\right] \\ &+\left[(p\cdot r)(q\cdot r)-r^2(p\cdot q)\right] \\ &\times\left[\eta_{00}(p,r;m^2)-\eta_{00}(s,-t;m^2)\right] \\ &+\left[(p\cdot q)(q\cdot r)-q^2(p\cdot r)\right] \\ &\times\left[\eta_{00}(p,q;m^2)-\eta_{00}(u,t;m^2)\right] \\ &+\Big[p^2\left(q^2r^2-(q\cdot r)^2\right) \\ &+q^2\left((p\cdot r)(q\cdot r)-r^2(p\cdot q)\right) \\ &+r^2\left((p\cdot q)(q\cdot r)-q^2(p\cdot r)\right)\Big]\xi_{000}(p,q,r)\Big\}.\end{aligned} \tag{47}$$

Here we have $\xi_{010}(p,q,r)=\xi_{100}(q,p,r)$ and $\xi_{001}(p,q,r)=\xi_{100}(r,q,p)$.

(iii) Functions η_{nml}:

$$\begin{aligned}\eta_{000}(p,q,r;m^2) &= \frac{1}{3}\left[\eta_{10}(s,-t;m^2)+\eta_{01}(s,-t;m^2)+\eta_{10}(s,u;m^2)\right] \\ &-\frac{2}{3}\left[\frac{1}{6}+m^2\xi_{000}(p,q,r)\right]+\frac{1}{3}p^2\xi_{100}(p,q,r) \\ &+\frac{1}{3}q^2\xi_{010}(p,q,r) \\ &+\frac{1}{3}r^2\xi_{001}(p,q,r),\end{aligned} \tag{48}$$

$$\begin{aligned}\eta_{100}(p,q,r;m^2) &= \frac{1}{4}\left[\eta_{10}(s,-t;m^2)-\eta_{11}(s,-t;m^2)\right. \\ &\left.+\eta_{10}(u,t;m^2)-\eta_{11}(u,t;m^2)-\eta_{20}(u,t;m^2)\right] \\ &-\frac{1}{2}\left[\frac{1}{24}+m^2\xi_{100}(p,q,r)\right]+\frac{p^2}{4}\xi_{200}(p,q,r) \\ &+\frac{q^2}{4}\xi_{110}(p,q,r)+\frac{r^2}{4}\xi_{101}(p,q,r).\end{aligned} \tag{49}$$

Again we can note that

$$\eta_{010}(p,q,r;m^2) = \eta_{100}(q,p,r;m^2),$$

and $\eta_{001}(p,q,r;m^2) = \eta_{100}(r,q,p;m^2)$.

The systematization obtained through the functions $\zeta_{nml}(p,q,r)$, $\xi_{nml}(p,q,r)$ and $\eta_{nml}(p,q,r;m^2)$ is enough to write all four-point amplitudes. In order to verify the Ward identities some properties of these functions are useful too. For the present purposes the following properties are sufficient:

(i) $n+m+l=1$:

$$(r\cdot p)\,\zeta_{100}(p,q,r) + (r\cdot q)\,\zeta_{010}(p,q,r) + r^2\zeta_{001}(p,q,r) = \frac{1}{2}\xi_{00}(u,t) - \frac{1}{2}\xi_{00}(p,q) + \frac{1}{2}r^2\zeta_{000}(p,q,r), \tag{50}$$

$$(p\cdot r)\,\xi_{100}(p,q,r) + (q\cdot r)\,\xi_{010}(p,q,r) + r^2\xi_{001}(p,q,r) = -\frac{1}{2}\eta_{00}\left(u,t;m^2\right) + \frac{1}{2}\eta_{00}\left(p,q;m^2\right) + \frac{1}{2}r^2\xi_{000}(p,q,r). \tag{51}$$

(ii) $n+m+l=2$:

$$(r\cdot p)\,\zeta_{200}(p,q,r) + (r\cdot q)\,\zeta_{110}(p,q,r) + r^2\zeta_{101}(p,q,r) = \frac{1}{2}\xi_{10}(u,t) - \frac{1}{2}\xi_{10}(p,q) + \frac{1}{2}r^2\zeta_{100}(p,q,r), \tag{52}$$

$$(r\cdot q)\,\zeta_{020}(p,q,r) + (r\cdot p)\,\zeta_{110}(p,q,r) + r^2\zeta_{011}(p,q,r) = \frac{1}{2}\xi_{01}(u,t) - \frac{1}{2}\xi_{01}(p,q) + \frac{1}{2}r^2\zeta_{010}(p,q,r), \tag{53}$$

$$r^2\zeta_{002}(p,q,r) + (r\cdot q)\,\zeta_{011}(p,q,r) + (r\cdot p)\,\zeta_{101}(p,q,r) = \frac{1}{2}\xi_{00}(u,t) - \frac{1}{2}\xi_{10}(u,t) - \frac{1}{2}\xi_{01}(u,t) - \frac{1}{2}\xi_{000}(p,q,r) + \frac{1}{2}r^2\zeta_{001}(p,q,r), \tag{54}$$

$$r^2\xi_{101}(p,q,r) + (r\cdot q)\,\xi_{110}(p,q,r) + (r\cdot p)\,\xi_{200}(p,q,r) = -\frac{1}{2}\eta_{10}\left(u,t;m^2\right) + \frac{1}{2}\eta_{10}\left(p,q;m^2\right) + \frac{1}{2}r^2\xi_{100}(p,q,r), \tag{55}$$

$$r^2\xi_{002}(p,q,r) + (r\cdot q)\,\xi_{011}(p,q,r) + (r\cdot p)\,\xi_{101}(p,q,r) = -\frac{1}{2}\eta_{00}\left(u,t;m^2\right) + \frac{1}{2}\eta_{10}\left(u,t;m^2\right) + \frac{1}{2}\eta_{01}\left(u,t;m^2\right) + \frac{1}{2}\eta_{000}\left(p,q,r;m^2\right) + \frac{1}{2}r^2\xi_{001}(p,q,r), \tag{56}$$

$$r^2\xi_{011}(p,q,r) + (r\cdot q)\,\xi_{020}(p,q,r) + (r\cdot p)\,\xi_{110}(p,q,r) = -\frac{1}{2}\eta_{01}\left(u,t;m^2\right) + \frac{1}{2}\eta_{01}\left(p,q;m^2\right) + \frac{1}{2}r^2\xi_{010}(p,q,r). \tag{57}$$

(iii) $n+m+l=3$:

$$(p\cdot r)\,\zeta_{300}(p,q,r) + (q\cdot r)\,\zeta_{210}(p,q,r) + r^2\zeta_{201}(p,q,r) = \frac{1}{2}\xi_{20}(u,t) - \frac{1}{2}\xi_{20}(p,q) + \frac{1}{2}r^2\zeta_{200}(p,q,r), \tag{58}$$

$$(q\cdot r)\,\zeta_{030}(p,q,r) + (p\cdot r)\,\zeta_{120}(p,q,r) + r^2\zeta_{021}(p,q,r) = \frac{1}{2}\xi_{02}(u,t) - \frac{1}{2}\xi_{02}(p,q) + \frac{1}{2}r^2\zeta_{020}(p,q,r), \tag{59}$$

$$r^2\zeta_{003}(p,q,r) + (p\cdot r)\,\zeta_{102}(p,q,r) + (q\cdot r)\,\zeta_{012}(p,q,r) = \frac{1}{2}\xi_{00}(u,t) - \xi_{10}(u,t) - \xi_{01}(u,t) + \frac{1}{2}\xi_{20}(u,t) + \xi_{11}(u,t) + \frac{1}{2}\xi_{02}(u,t) - \xi_{001}(p,q,r) + \frac{1}{2}r^2\zeta_{002}(p,q,r), \tag{60}$$

$$(p\cdot r)\,\zeta_{210}(p,q,r) + (q\cdot r)\,\zeta_{120}(p,q,r) + r^2\zeta_{111}(p,q,r) = \frac{1}{2}\xi_{11}(u,t) - \frac{1}{2}\xi_{11}(p,q) + \frac{1}{2}r^2\zeta_{110}(p,q,r), \tag{61}$$

$$(p\cdot r)\,\zeta_{201}(p,q,r) + (q\cdot r)\,\zeta_{111}(p,q,r) + r^2\zeta_{102}(p,q,r) = \frac{1}{2}\xi_{10}(u,t) - \frac{1}{2}\xi_{20}(u,t) - \frac{1}{2}\xi_{11}(u,t) - \frac{1}{2}\xi_{100}(p,q,r) + \frac{1}{2}r^2\zeta_{101}(p,q,r), \tag{62}$$

$$(p\cdot r)\,\zeta_{111}(p,q,r) + (q\cdot r)\,\zeta_{021}(p,q,r) + r^2\zeta_{012}(p,q,r) = \frac{1}{2}\xi_{01}(u,t) - \frac{1}{2}\xi_{11}(u,t) - \frac{1}{2}\xi_{02}(u,t) - \frac{1}{2}\xi_{010}(p,q,r) + \frac{1}{2}r^2\zeta_{011}(p,q,r). \tag{63}$$

(iv) $n+m+l=4$:

$$r^2\zeta_{004}(p,q,r) + (r\cdot p)\,\zeta_{103}(p,q,r)$$

$$+\left(r\cdot q\right)\zeta_{013}\left(p,q,r\right)$$
$$=\frac{1}{2}\xi_{00}\left(u,t\right)-\frac{3}{2}\xi_{10}\left(u,t\right)-\frac{3}{2}\xi_{01}\left(u,t\right)+\frac{3}{2}\xi_{20}\left(u,t\right)$$
$$+3\xi_{11}\left(u,t\right)+\frac{3}{2}\xi_{02}\left(u,t\right)-\frac{1}{2}\xi_{30}\left(u,t\right)-\frac{3}{2}\xi_{21}\left(u,t\right)$$
$$-\frac{3}{2}\xi_{12}\left(u,t\right)-\frac{1}{2}\xi_{03}\left(u,t\right)-\frac{3}{2}\xi_{002}\left(p,q,r\right)$$
$$+\frac{r^2}{2}\zeta_{003}\left(p,q,r\right), \tag{64}$$

$$r^2\zeta_{103}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{112}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{202}\left(p,q,r\right)$$
$$=\frac{1}{2}\xi_{10}\left(u,t\right)-\xi_{20}\left(u,t\right)-\xi_{11}\left(u,t\right)+\frac{1}{2}\xi_{30}\left(u,t\right)$$
$$+\xi_{21}\left(u,t\right)+\frac{1}{2}\xi_{12}\left(u,t\right)-\xi_{101}\left(p,q,r\right)$$
$$+\frac{r^2}{2}\zeta_{102}\left(p,q,r\right), \tag{65}$$

$$r^2\zeta_{022}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{031}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{121}\left(p,q,r\right)$$
$$=\frac{1}{2}\xi_{02}\left(u,t\right)-\frac{1}{2}\xi_{12}\left(u,t\right)-\frac{1}{2}\xi_{03}\left(u,t\right)$$
$$-\frac{1}{2}\xi_{020}\left(p,q,r\right)+\frac{r^2}{2}\zeta_{021}\left(p,q,r\right), \tag{66}$$

$$r^2\zeta_{013}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{022}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{112}\left(p,q,r\right)$$
$$=\frac{1}{2}\xi_{01}\left(u,t\right)-\xi_{11}\left(u,t\right)-\xi_{02}\left(u,t\right)+\frac{1}{2}\xi_{21}\left(u,t\right)$$
$$+\xi_{12}\left(u,t\right)+\frac{1}{2}\xi_{03}\left(u,t\right)-\xi_{011}\left(p,q,r\right)$$
$$+\frac{r^2}{2}\zeta_{012}\left(p,q,r\right), \tag{67}$$

$$r^2\zeta_{112}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{121}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{211}\left(p,q,r\right)$$
$$=\frac{1}{2}\xi_{11}\left(u,t\right)-\frac{1}{2}\xi_{21}\left(u,t\right)-\frac{1}{2}\xi_{12}\left(u,t\right)$$
$$-\frac{1}{2}\xi_{110}\left(p,q,r\right)+\frac{r^2}{2}\zeta_{111}\left(p,q,r\right), \tag{68}$$

$$r^2\zeta_{301}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{310}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{400}\left(p,q,r\right)$$
$$=\frac{1}{2}\xi_{30}\left(u,t\right)-\frac{1}{2}\xi_{30}\left(p,q\right)+\frac{r^2}{2}\zeta_{300}\left(p,q,r\right), \tag{69}$$

$$r^2\zeta_{031}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{040}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{130}\left(p,q,r\right) \tag{70}$$
$$=\frac{1}{2}\xi_{03}\left(u,t\right)-\frac{1}{2}\xi_{03}\left(p,q\right)+\frac{1}{2}r^2\zeta_{030}\left(p,q,r\right),$$

$$r^2\zeta_{211}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{220}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{310}\left(p,q,r\right) \tag{71}$$
$$=\frac{1}{2}\xi_{21}\left(u,t\right)-\frac{1}{2}\xi_{21}\left(p,q\right)+\frac{1}{2}r^2\zeta_{210}\left(p,q,r\right),$$

$$r^2\zeta_{202}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{211}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{301}\left(p,q,r\right)$$
$$=\frac{1}{2}\xi_{20}\left(u,t\right)-\frac{1}{2}\xi_{30}\left(u,t\right)-\frac{1}{2}\xi_{21}\left(u,t\right)$$
$$-\frac{1}{2}\xi_{200}\left(p,q,r\right)+\frac{1}{2}r^2\zeta_{201}\left(p,q,r\right), \tag{72}$$

$$r^2\zeta_{121}\left(p,q,r\right)+\left(r\cdot q\right)\zeta_{130}\left(p,q,r\right)$$
$$+\left(r\cdot p\right)\zeta_{220}\left(p,q,r\right) \tag{73}$$
$$=\frac{1}{2}\xi_{12}\left(u,t\right)-\frac{1}{2}\xi_{12}\left(p,q\right)+\frac{1}{2}r^2\zeta_{120}\left(p,q,r\right),$$

Similar relations can be obtained for other components of the set by exploring the properties relating these functions which are the interchanges $p\leftrightarrow q$, $p\leftrightarrow r$, and $q\leftrightarrow r$ (analogously to the ξ_{nm} functions). The systematization allows us to treat the perturbative four-point amplitudes in an exact way. Let us now consider the evaluation of the integrals (1)–(4) in terms of the systematization introduced.

5 Manipulations and calculations of the Feynman integrals

Let us now evaluate the Feynman integrals defined in Sect. 2 following the procedure introduced in Sect. 3. For each Feynman integral we first separate the finite and divergent parts and then, by solving the finite integrals, we put the result in terms of the basic divergent objects and basic divergent structure functions previously defined.

5.1 One-point Feynman integrals

We start by the one having the highest degree of divergence, which, after choosing $N=3$ in the expression (6), can be written as

$$I_{1\mu}\left(k_1\right)=-\int_\Lambda\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{2k_\mu\left(k_1\cdot k\right)}{\left[P\left(0,m\right)\right]^2}$$
$$+k_1^\nu k_1^\alpha k_1^\beta\left\{\int_\Lambda\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4g_{\alpha\beta}k_\mu k_\nu}{\left[P\left(0,m\right)\right]^3}\right.$$
$$\left.-\int_\Lambda\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{8k_\alpha k_\beta k_\mu k_\nu}{\left[P\left(0,m\right)\right]^4}\right\}$$
$$-\left\{\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{6k_1^4\left(k_1\cdot k\right)k_\mu}{\left[P\left(0,m\right)\right]^4}\right.$$
$$\left.-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(k_1^2+2k_1\cdot k\right)^4k_\mu}{\left[P\left(0,m\right)\right]^4\left[P\left(k_1,m\right)\right]}\right\}. \tag{74}$$

Note the absence of integrals having odd integrands as a consequence of the even character for our implicit regulator and note that we have removed the subscript Λ on the two last integrals as a consequence of the connection limit requirement. The finite integrals this way obtained can be solved by standard techniques without any restriction. The result is an exact cancellation between them. With this information, we organize the divergent parts remaining in terms of the objects defined in (7)–(11). We write then

$$\begin{aligned} I_{1\mu}(k_1) &= -k_{1\mu}\left[I_{\text{quad}}(m^2)\right] - k_1^\beta\left[\nabla_{\beta\mu}\right] - \frac{1}{3}k_1^\beta k_1^\alpha k_1^\nu\left[\Box_{\alpha\beta\mu\nu}\right] \\ &\quad -\frac{1}{3}k_{1\mu}k_1^\alpha k_1^\beta\left[\Delta_{\alpha\beta}\right] + \frac{1}{3}k_1^2 k_1^\nu\left[\Delta_{\mu\nu}\right]. \end{aligned} \tag{75}$$

It is important to note that this result is still compatible with any regularization technique. If all the integration is changed to 2ω dimensions we get the result which we could find by the DR technique at this stage. The remaining steps, i.e., the value of the standard divergent objects, need to be evaluated according to that technique. If we want to use 4D regularization, like the Pauli–Villars or the sharp cutoff one, all divergent objects present in the expression above need to be evaluated in the specific point of view of the particular philosophy adopted. Another aspect which is important to emphasize is that there is no assumption about the ambiguities at the point we arrived in our calculation. The choices we need to adopt, which are arbitrary once they are not dictated by Feynman rules, become, in principle, necessary only after our final result. We need a specific regularization method only when we want to attribute a value to the basic objects in (7)–(11) and different philosophies will differ only by these results. In particular, the objects $\Box$, Δ and ∇ are obtained exactly zero in the DR technique, and different from zero in the sharp cutoff regularization [2]. In this work the discussion of the value for the divergent objects does not play an important role. Our intention is precisely to generate the results in a way that they can be used in a posterior step by all reasonable regularizations.

For the quadratic divergent integral in (1) we apply the same recipe. Choosing in (6) the value $N=2$ we get first

$$\begin{aligned} I_1(k_1) &= \int_\Lambda \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{[P(0,m)]} \\ &+k_1^\mu k_1^\nu\left\{\int_\Lambda \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4k_\mu k_\nu}{[P(0,m)]^3} - \int_\Lambda \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{[P(0,m)]^2}\right\} \\ &+\left\{\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_1^4}{[P(0,m)]^3} - \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)^3}{[P(0,m)]^3[P(k_1,m)]}\right\}. \end{aligned} \tag{76}$$

Again, we dropped an odd term and the subscript Λ on the last two terms, which integration produces an exact cancellation. Then the result is

$$I_1(k_1) = \left[I_{\text{quad}}(m^2)\right] + k_1^\mu k_1^\nu\left[\Delta_{\mu\nu}\right]. \tag{77}$$

5.2 Two-point Feynman integrals

Let us now consider the integrals having two propagators. First the simplest one; the I_2 integral which, following our strategy, can be written as

$$\begin{aligned} I_2(k_1,k_2) &= \int_\Lambda \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{[P(0,m)]^2} \\ &\quad -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)}{[P(0,m)]^2[P(k_1,m)]} \\ &\quad -\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k_2^2+2k_2\cdot k)}{[P(0,m)]^2[P(k_2,m)]} \\ &\quad +\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)(k_2^2+2k_2\cdot k)}{[P(0,m)]^2[P(k_1,m)][P(k_2,m)]}, \end{aligned} \tag{78}$$

where we have chosen, in (6), $N=1$ for the two denominators involved, which is very convenient, although not unique, since it maintains the symmetry in k_1 and k_2. The divergent content of this integral is present in the basic divergent object $I_{\log}(m^2)$. The remaining integrals are finite and yield

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k_i^2+2k_i\cdot k)}{[P(0,m)]^2[P(k_i,m)]} = \mathrm{i}\,(4\pi)^{-2}Z_0\left(k_i^2;m^2\right), \tag{79}$$

$$\begin{aligned} &\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot k)(k_2^2+2k_2\cdot k)}{[P(0,m)]^2[P(k_1,m)][P(k_2,m)]} \\ &= \mathrm{i}\,(4\pi)^{-2}\left[Z_0\left(k_1^2;m^2\right)+Z_0\left(k_2^2;m^2\right)-Z_0\left(p^2;m^2\right)\right], \end{aligned} \tag{80}$$

where we have identified the basic functions for two-point structures defined in (12) and defined the external momentum $p=k_2-k_1$. Collecting the results, the logarithmically divergent integral can be written as

$$I_2(p) = I_{\log}\left(m^2\right) - \mathrm{i}\,(4\pi)^{-2}Z_0\left(p^2;m^2\right). \tag{81}$$

In order to evaluate $I_{2\mu}(k_1,k_2)$ we first use the identity (6) taking $N=1$ for both denominators on the integral, to obtain

$$\begin{aligned} I_{2\mu}(k_1,k_2) &= -\frac{1}{2}(k_1+k_2)^\alpha\int_\Lambda \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4k_\alpha k_\mu}{[P(0,m)]^3} \\ &\quad +\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(k_1^2+2k_1\cdot\ k)^2k_\mu}{[P(0,m)]^3[P(k_1,m)]} \end{aligned}$$

$$+\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(k_2^2+2k_2\cdot\ k)^2 k_\mu}{[P(0,m)]^3 [P(k_2,m)]} \tag{82}$$
$$+\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(k_1^2+2k_1\cdot k)(k_2^2+2k_2\cdot k)k_\mu}{[P(0,m)]^2 [P(k_1,m)][P(k_2,m)]}.$$

Only an integral having an odd integrand has been removed on the right hand side of the above equation, in addition to the subscript Λ on the last three (finite) integrals, which, after the integration, produce the results

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(k_i^2+2k_1\cdot\ k)^2 k_\mu}{[P(0,m)]^3 [P(k_i,m)]}$$
$$= \mathrm{i}(4\pi)^{-2} k_{i\mu} Z_1(k_1^2;m^2), \tag{83}$$
$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{(k_1^2+2k_1\cdot k)(k_2^2+2k_2\cdot k)k_\mu}{[P(0,m)]^2 [P(k_1,m)][P(k_2,m)]}$$
$$= -\mathrm{i}(4\pi)^{-2}\left[k_{1\mu}Z_1(k_1^2;m^2) + k_{2\mu}Z_1(k_2^2;m^2) - P_\mu Z_1(p^2;m^2)\right]. \tag{84}$$

Here $P = k_1 + k_2$ is an ambiguous combination of the internal arbitrary momenta. If we also consider the property (19) we can write the result as

$$I_{2\mu}(k_1,k_2) = -\frac{1}{2}P^\rho[\Delta_{\mu\rho}] - \frac{1}{2}P_\mu\left[I_{\log}(m^2)\right] + \mathrm{i}(4\pi)^{-2}\frac{1}{2}P_\mu Z_0(p^2;m^2). \tag{85}$$

Following strictly the same procedure as adopted on the evaluation of the logarithmically and linearly divergent cases, choosing adequate values for N in the identity (6), after a long and tedious calculation, we get

$$\begin{aligned}
&I_{2\mu\nu}(k_1,k_2)\\
&= \frac{1}{2}[\nabla_{\mu\nu}] - \frac{1}{12}p^2[\Delta_{\mu\nu}]\\
&+\frac{1}{6}\left(k_2^\alpha k_2^\beta + k_1^\alpha k_2^\beta + k_1^\alpha k_1^\beta\right)[\Box_{\alpha\beta\mu\nu}]\\
&+\frac{1}{6}(g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha})(k_2^\alpha k_2^\rho + k_1^\alpha k_2^\rho + k_1^\alpha k_1^\rho)[\Delta_{\beta\rho}]\\
&+\frac{1}{2}g_{\mu\nu}\left[I_{\text{quad}}(m^2)\right] - \frac{1}{12}g_{\mu\nu}p^2\left[I_{\log}(m^2)\right]\\
&+\frac{1}{6}(2k_{2\nu}k_{2\mu}+k_{1\nu}k_{2\mu}+k_{1\mu}k_{2\nu}+2k_{1\nu}k_{1\mu})\left[I_{\log}(m^2)\right]\\
&+\mathrm{i}(4\pi)^{-2}\left\{-\frac{1}{4}P_\mu P_\nu Z_0(p^2;m^2)\right. \qquad (86)\\
&\left.+(p_\mu p_\nu - g_{\mu\nu}p^2)\left[-Z_2(p^2;m^2)+\frac{1}{4}Z_0(p^2;m^2)\right]\right\}.
\end{aligned}$$

Note that by using the relations (19) and (20) we can eliminate $Z_2(p^2;m^2)$ in terms of $Z_0(p^2;m^2)$ which represents, in our language, the reduction of the tensor integral to that scalar.

5.3 Three-point Feynman integrals

Now we evaluate the integrals with three propagators. The first of them is finite and may be directly calculated. We write the results as

$$I_3(p,q) = \mathrm{i}(4\pi)^{-2}\xi_{00}(p,q), \tag{87}$$

where we maintained $p = k_2 - k_1$, introduced the definition $q = k_3 - k_1$, and used the definition of the $\xi_{nm}(p,q)$ functions. The second integral we consider is finite too so that it can be directly integrated by using standard techniques. The result can be written as

$$\begin{aligned}
&I_{3\mu}(p,q)\\
&= -\mathrm{i}(4\pi)^{-2}[p_\mu\xi_{01}(p,q) + q_\mu\xi_{10}(p,q)\\
&\quad + k_{1\mu}\xi_{00}(p,q)]. \qquad (88)
\end{aligned}$$

The expression above represents an excellent opportunity to make a comment. Given the results (24) and (25), the equation above can be written in the form

$$\begin{aligned}
&\mathrm{i}(4\pi)^2 I_{3\mu}(p,q)\\
&= \frac{p_\mu}{2}C_1\left\{\left(\frac{p\cdot q - p^2}{p^2q^2}\right)Z_0\left((p-q)^2;m^2\right)\right.\\
&-\frac{p\cdot q}{p^2q^2}Z_0(q^2;m^2) + \frac{1}{q^2}Z_0(p^2;m^2)\\
&\left.+\left(\frac{q^2-p\cdot q}{q^2}\right)\xi_{00}(p,q)\right\}\\
&+\frac{q_\mu}{2}C_1\left\{\left(\frac{p\cdot q - q^2}{p^2q^2}\right)Z_0\left((p-q)^2;m^2\right)\right.\\
&-\frac{p\cdot q}{p^2q^2}Z_0(p^2;m^2) + \frac{1}{p^2}Z_0(q^2;m^2)\\
&\left.+\left(\frac{p^2-p\cdot q}{p^2}\right)\xi_{00}(p,q)\right\}\\
&+k_{1\mu}\xi_{00}(p,q). \qquad (89)
\end{aligned}$$

The expression above can be easily identified with the a typical form of results obtained through the Passarino–Veltman procedure. If the function $\xi_{00}(p,q)$ is identified with the scalar integral I_3, the above expression can be viewed as a reduction of a vector integral to the corresponding scalar one. The $Z_k(p^2;m^2)$ functions obtained above are the finite parts of the two denominators integrals resulting from the cancellation of one of such terms when the scalar product of an external momenta with the integrating momentum k is eliminated. Here no divergent integral has to be solved. We put the result in the form (89) by using properties of $\xi_{nm}(p,q)$ functions.

The next integral of the set (3), which is $I_{3\mu\nu}(k_1,k_2)$, is logarithmically divergent and we need to rewrite the integrand in the first step. We initially have

$$
\begin{aligned}
&I_{3\mu\nu}\left(k_1,k_2,k_3\right)\\
&=\int_\Lambda \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\nu k_\mu}{\left[P\left(0,m\right)\right]^3}\\
&\quad-\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(k_3^2+2k\cdot k_3\right)k_\nu k_\mu}{\left[P\left(0,m\right)\right]^3\left[P\left(k_3,m\right)\right]}\\
&\quad-\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(k_2^2+2k\cdot k_2\right)k_\nu k_\mu}{\left[P\left(0,m\right)\right]^2\left[P\left(k_2,m\right)\right]\left[P\left(k_3,m\right)\right]}\\
&\quad-\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(k_1^2+2k\cdot k_1\right)k_\nu k_\mu}{\left[P\left(0,m\right)\right]\left[P\left(k_1,m\right)\right]\left[P\left(k_2,m\right)\right]}\\
&\qquad\times\frac{1}{\left[P\left(k_3,m\right)\right]}. \qquad (90)
\end{aligned}
$$

Solving the finite integrals we can put the results in the form

$$
\begin{aligned}
&I_{3\mu\nu}\left(k_1,k_2,k_3\right)\\
&=\frac{1}{4}\left(\Delta_{\mu\nu}\right)+\frac{1}{4}g_{\mu\nu}\left[I_{\log}\left(m^2\right)\right]\\
&\quad+\mathrm{i}\left(4\pi\right)^{-2}\left\{-\frac{1}{2}g_{\mu\nu}\eta_{00}\left(p,q;m^2\right)\right.\\
&\quad+p_\mu p_\nu\xi_{02}\left(p,q\right)+q_\mu q_\nu\xi_{20}\left(p,q\right)\\
&\quad\left.+\left(p_\mu q_\nu+q_\mu p_\nu\right)\xi_{11}\left(p,q\right)\right\}\\
&\quad-\left(g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}\right)\\
&\quad\times\left\{k_1^\alpha\left[I_3^\beta\left(p,q\right)\right]-\frac{1}{2}k_1^\alpha k_1^\beta\left[I_3\left(p,q\right)\right]\right\}. \qquad (91)
\end{aligned}
$$

Now let us consider the linearly divergent structure, the integral $I_{3\lambda\mu\nu}(k_1,k_2)$. The first step is to rewrite it using identity (6) as we did above and next we solve the finite integrals to write the result as

$$
\begin{aligned}
&I_{3\lambda\mu\nu}\left(k_1,k_2,k_3\right)\\
&=I'_{3\lambda\mu\nu}\left(p,q\right)+I'_{3\lambda\mu\ \nu}\left(q,p\right)\\
&\quad-\frac{1}{12}\left(k_1^\alpha+k_2^\alpha+k_3^\alpha\right)\left(\Box_{\alpha\lambda\mu\nu}\right)\\
&\quad-\frac{1}{12}\left(g_{\mu\alpha}g_{\nu\beta}g_{\lambda\gamma}+g_{\nu\alpha}g_{\lambda\beta}g_{\mu\gamma}+g_{\lambda\alpha}g_{\mu\beta}g_{\nu\gamma}\right)\\
&\quad\times\left(k_1^\alpha+k_2^\alpha+k_3^\alpha\right)\left(\Delta^{\beta\gamma}\right)\\
&\quad-\frac{1}{12}\left(g_{\lambda\alpha}g_{\mu\nu}+g_{\nu\alpha}g_{\mu\lambda}+g_{\mu\alpha}g_{\lambda\nu}\right)\left(k_1^\alpha+k_2^\alpha+k_3^\alpha\right)\\
&\quad\times\left[I_{\log}\left(m^2\right)\right]\\
&\quad-\left(g_{\mu\alpha}g_{\nu\beta}g_{\lambda\gamma}+g_{\nu\alpha}g_{\lambda\beta}g_{\mu\gamma}+g_{\lambda\alpha}g_{\mu\beta}g_{\nu\gamma}\right)\\
&\quad\times\left\{k_1^\alpha\left[I_3^{\beta\gamma}\left(p,q\right)\right]-k_1^\alpha k_1^\beta\left[I_3^\gamma\left(p,q\right)\right]\right.\\
&\quad\left.+\frac{1}{3}k_1^\alpha k_1^\beta k_1^\gamma\left[I_3\left(p,q\right)\right]\right\}, \qquad (92)
\end{aligned}
$$

with

$$
\begin{aligned}
&I'_{3\lambda\mu\nu}\left(p,q\right)\\
&=\mathrm{i}\left(4\pi\right)^{-2}\left\{\frac{1}{2}\left(g_{\mu\lambda}p_\nu+g_{\nu\lambda}p_\mu+g_{\mu\nu}p_\lambda\right)\eta_{01}\left(p,q;m^2\right)\right.\\
&\quad-\left(p_\lambda p_\mu q_\nu+p_\lambda q_\mu p_\nu+q_\lambda p_\mu p_\nu\right)\xi_{12}\left(p,q\right)\\
&\quad\left.-p_\lambda p_\mu p_\nu\xi_{03}\left(p,q\right)\right\}. \qquad (93)
\end{aligned}
$$

In fundamental gauge theories the considered integrals are enough to evaluate the one-loop amplitudes.

5.4 Four-point Feynman integrals

Finally, we consider the four-point function integrals. Only one of them is a divergent structure which makes the job easy. The first, the scalar one, can be written as

$$
I_4\left(p,q,r\right)=\mathrm{i}\left(4\pi\right)^{-2}\zeta_{000}\left(p,q,r\right), \qquad (94)
$$

where we have identified the four-point structure functions previously defined in (43) and also the external momentum $r=k_4-k_1$. Next, it is immediate to see that

$$
\begin{aligned}
&I_{4\mu}\left(p,q,r\right)\\
&=-\mathrm{i}\left(4\pi\right)^{-2}\left[p_\mu\zeta_{100}\left(p,q,r\right)+q_\mu\zeta_{010}\left(p,q,r\right)\right.\\
&\quad\left.+r_\mu\zeta_{001}\left(p,q,r\right)\right]-k_{1\mu}\left[I_4\left(p,q,r\right)\right], \qquad (95)
\end{aligned}
$$

and that with two Lorentz index becomes

$$
\begin{aligned}
&I_{4\mu\nu}\left(p,q,r\right)\\
&=I'_{4\mu\nu}\left(p,q,r\right)+I'_{4\mu\nu}\left(q,p,r\right)+I'_{4\mu\nu}\left(r,q,p\right)\\
&\quad-\left(g_{\mu\alpha}g_{\nu\beta}+g_{\nu\alpha}g_{\mu\beta}\right)\\
&\quad\times\left\{k_1^\alpha\left[I_4^\beta\left(p,q,r\right)\right]-\frac{1}{2}k_1^\alpha k_1^\beta\left[I_4\left(p,q,r\right)\right]\right\}, \qquad (96)
\end{aligned}
$$

with

$$
\begin{aligned}
&I'_{4\mu\nu}\left(p,q,r\right)\\
&=\mathrm{i}\left(4\pi\right)^{-2}\left\{\frac{1}{6}g_{\mu\nu}\xi_{000}\left(p,q,r\right)+p_\mu p_\nu\zeta_{200}\left(p,q,r\right)\right.\\
&\quad\left.+q_\mu r_\nu\zeta_{011}\left(p,q,r\right)+r_\mu p_\nu\zeta_{101}\left(p,q,r\right)\right\}. \qquad (97)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&I_{4\mu\nu\lambda}\left(p,q,r\right)\\
&=I'_{4\mu\nu\lambda}\left(p,q,r\right)+I'_{4\mu\nu\ \lambda}\left(q,p,r\right)+I'_{4\mu\nu\lambda}\left(r,q,p\right)\\
&\quad-\left(g_{\mu\alpha}g_{\nu\beta}g_{\lambda\gamma}+g_{\nu\alpha}g_{\lambda\beta}g_{\mu\gamma}+g_{\lambda\alpha}g_{\mu\beta}g_{\nu\gamma}\right)\\
&\quad\times\left\{k_1^\alpha\left[I_4^{\beta\gamma}\left(p,q,r\right)\right]-k_1^\alpha k_1^\beta\left[I_4^\gamma\left(p,q,r\right)\right]\right.
\end{aligned}
$$

$$+\frac{1}{3}k_1^{\alpha}k_1^{\beta}k_1^{\gamma}\left[I_3\left(p,q,r\right)\right]\Big\}, \quad (98)$$

where

$$\begin{aligned}
&I'_{4\mu\nu\lambda}\left(p,q,r\right)\\
&=-\mathrm{i}\left(4\pi\right)^{-2}\\
&\quad\times\left\{\frac{1}{2}\left(g_{\mu\nu}p_{\lambda}+g_{\mu\lambda}p_{\nu}+g_{\nu\lambda}p_{\mu}\right)\xi_{100}\left(p,q,r\right)\right.\\
&\quad+p_{\mu}p_{\nu}p_{\lambda}\zeta_{300}\left(p,q,r\right)\\
&\quad+\left(p_{\mu}p_{\nu}q_{\lambda}+p_{\mu}q_{\nu}p_{\lambda}+q_{\mu}p_{\nu}p_{\lambda}\right)\zeta_{210}\left(p,q,r\right)\\
&\quad+\left(p_{\mu}p_{\nu}r_{\lambda}+p_{\mu}r_{\nu}p_{\lambda}+r_{\mu}p_{\nu}p_{\lambda}\right)\zeta_{201}\left(p,q,r\right)\\
&\quad\left.+\left(p_{\mu}q_{\nu}r_{\lambda}+p_{\mu}r_{\nu}q_{\lambda}\right)\zeta_{111}\left(p,q,r\right)\right\}. \quad (99)
\end{aligned}$$

The last one we consider is the logarithmically divergent one, which we write as

$$\begin{aligned}
&I_{4\mu\nu\alpha\beta}\left(p,q,r\right)\\
&=\frac{1}{24}\left\{\left[\Box_{\alpha\beta\mu\nu}\right]+g_{\alpha\beta}\left[\Delta_{\mu\nu}\right]+g_{\alpha\nu}\left[\Delta_{\mu\beta}\right]+g_{\alpha\mu}\left[\Delta_{\beta\nu}\right]\right\}\\
&\quad+\frac{1}{24}\left(g_{\alpha\beta}g_{\mu\nu}+g_{\alpha\mu}g_{\beta\nu}+g_{\alpha\nu}g_{\mu\beta}\right)\left[I_{\log}\left(m^2\right)\right]\\
&\quad+I'_{4\mu\nu\alpha\beta}\left(p,q,r\right)+I'_{4\mu\nu\alpha\beta}\left(q,p,r\right)+I'_{4\mu\nu\alpha\ \beta}\left(r,q,p\right)\\
&\quad-\left(g_{\alpha\rho}g_{\beta\gamma}g_{\mu\tau}g_{\nu\lambda}+g_{\beta\rho}g_{\nu\gamma}g_{\alpha\tau}g_{\mu\lambda}+g_{\mu\rho}g_{\beta\gamma}g_{\nu\tau}g_{\alpha\lambda}\right.\\
&\quad\left.+g_{\nu\rho}g_{\alpha\gamma}g_{\mu\tau}g_{\beta\lambda}\right)\\
&\quad\times\left\{k_1^{\rho}\left[I_4^{\gamma\tau\lambda}\left(p,q,r\right)\right]-\frac{1}{2}k_1^{\rho}k_1^{\gamma}\left[I_4^{\tau\lambda}\left(p,q,r\right)\right]\right. \quad (100)\\
&\quad-\frac{1}{2}k_1^{\rho}k_1^{\tau}\left[I_4^{\gamma\lambda}\left(p,q,r\right)\right]-\frac{1}{2}k_1^{\rho}k_1^{\lambda}\left[I_4^{\tau\gamma}\left(p,q,r\right)\right]\\
&\quad\left.+k_1^{\rho}k_1^{\gamma}k_1^{\tau}\left[I_4^{\lambda}\left(p,q,r\right)\right]-k_1^{\rho}k_1^{\gamma}k_1^{\tau}k_1^{\lambda}\left[I_4\left(p,q,r\right)\right]\right\},
\end{aligned}$$

where

$$\begin{aligned}
&I'_{4\mu\nu\alpha\beta}\left(p,q,r\right)\\
&=I''_{4\mu\nu\alpha\beta}\left(p,q,r\right)+I''_{4\nu\mu\alpha\beta}\left(p,q,r\right)+I''_{4\beta\nu\alpha\ \mu}\left(p,q,r\right)\\
&I''_{4\mu\nu\alpha\beta}\left(p,q,r\right)\\
&=\mathrm{i}\left(4\pi\right)^{-2}\left\{-\frac{1}{12}g_{\alpha\mu}g_{\beta\nu}\eta_{000}\left(p,q,r;m^2\right)\right.\\
&\quad+\frac{1}{2}\left(g_{\mu\alpha}p_{\nu}p_{\beta}+g_{\nu\beta}p_{\mu}p_{\alpha}\right)\xi_{200}\left(p,q,r\right)\\
&\quad+\frac{1}{2}\left[g_{\mu\alpha}\left(q_{\nu}r_{\beta}+r_{\nu}q_{\beta}\right)+g_{\nu\beta}\left(q_{\mu}r_{\alpha}+r_{\mu}q_{\alpha}\right)\right]\\
&\quad\times\xi_{011}\left(p,q,r\right)\\
&\quad+\frac{1}{3}p_{\mu}p_{\nu}p_{\alpha}p_{\beta}\zeta_{400}\left(p,q,r\right)\\
&\quad+\left(\frac{1}{3}r_{\alpha}q_{\mu}q_{\nu}q_{\beta}+r_{\mu}q_{\nu}q_{\alpha}q_{\beta}\right)\zeta_{031}\left(p,q,r\right)\\
&\quad+\left(\frac{1}{3}q_{\alpha}r_{\mu}r_{\nu}r_{\beta}+q_{\mu}r_{\nu}r_{\alpha}r_{\beta}\right)\zeta_{013}\left(p,q,r\right)\\
&\quad+\left(q_{\mu}r_{\nu}q_{\alpha}r_{\beta}+r_{\mu}q_{\nu}r_{\alpha}q_{\beta}\right)\zeta_{022}\left(p,q,r\right)\\
&\quad+\left(p_{\mu}q_{\nu}p_{\alpha}r_{\beta}+p_{\mu}r_{\nu}p_{\alpha}q_{\beta}+q_{\mu}p_{\nu}r_{\alpha}p_{\beta}+r_{\mu}p_{\nu}q_{\alpha}p_{\beta}\right)\\
&\quad\left.\times\zeta_{211}\left(p,q,r\right)\right\}. \quad (101)
\end{aligned}$$

With the above results for the Feynman integrals at hand we can perform all the one-loop amplitudes for one, two, three and four fermionic propagators in the context of fundamental gauge theories. In the next section we evaluate some representative amplitudes involving vector vertices.

6 Physical amplitudes

In the preceding sections we have considered the evaluation of the Feynman integrals introduced in Sect. 2, which are crucial for the one-loop calculation in the context of fundamental gauge theories like QED. All the integrals have been written in terms of the set of divergent objects; $\Box_{\alpha\beta\mu\nu}$, $\Delta_{\mu\nu}$, $\nabla_{\mu\nu}$, $I_{\log}(m^2)$, and $I_{\mathrm{quad}}(m^2)$, defined in (7)–(11) and in terms of the functions $Z_k(p^2;m^2)$, $\xi_{nm}(p,q)$, and $\zeta_{nml}(p,q,r)$ defined in (12), (22), and (43) for two-, three- and four-point functions respectively. By using properties relating the above cited functions, all one-loop amplitudes can be reduced to a combination of only three basic pieces: $Z_0(p^2;m^2)$, $\xi_{00}(p,q)$, and $\zeta_{000}(p,q,r)$. In the present section we evaluate some representative amplitudes of the perturbative calculations by using the systematization introduced in the preceding sections. We consider an example for each number of points taking the amplitude corresponding to the highest degree of divergence. With this attitude we have an opportunity to use all the ingredients we have introduced in our proposed systematization. In the next section we consider the relations among Green's functions, ambiguities and Ward identities. First we adopt the very general definition for such structures.

(I) One-point functions:

$$T^{i_1}\left(k_1\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left\{\Gamma_{i_1}\frac{1}{P'\left(\not k_1,m\right)}\right\}, \quad (102)$$

(II) Two-point functions:

$$\begin{aligned}
&T^{i_1i_2}\left(k_1,k_2\right) \quad (103)\\
&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left\{\Gamma_{i_1}\frac{1}{P'\left(\not k_1,m\right)}\Gamma_{i_2}\frac{1}{P'\left(\not k_2,m\right)}\right\},
\end{aligned}$$

(III) Three-point functions:

$$\begin{aligned}
&T^{i_1i_2i_3}\left(k_1,k_2,k_3\right)\\
&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left\{\Gamma_{i_1}\frac{1}{P'\left(\not k_1,m\right)}\right.\\
&\quad\left.\times\Gamma_{i_2}\frac{1}{P'\left(\not k_2,m\right)}\Gamma_{i_3}\frac{1}{P'\left(\not k_3,m\right)}\right\}. \quad (104)
\end{aligned}$$

(IV) Four-point functions:

$$T^{i_1 i_2 i_3 i_4}(k_1,k_2,k_3,k_4) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left\{ \Gamma_{i_1} \frac{1}{P'(\not k_1,m)} \times \Gamma_{i_2} \frac{1}{P'(\not k_2,m)} \Gamma_{i_3} \frac{1}{P'(\not k_3,m)} \Gamma_{i_4} \frac{1}{P'(\not k_4,m)} \right\}, \tag{105}$$

where

$$[P'(\not k_i,m)]^{-1} = (\not k + \not k_i) - m$$

is the spin 1/2 free-fermion propagator and $\Gamma_S = 1$, $\Gamma_P = \gamma_5$, $\Gamma_V = \gamma_\mu$, $\Gamma_A = \gamma_\mu\gamma_5$, and $\Gamma_T = \sigma_{\mu\nu}$ are the vertices. Now we consider particular examples belonging to the general set of amplitudes defined above.

6.1 The vector one-point function

We start considering the evaluation of the Green's function containing only one fermionic propagator and the vector vertex operator $\Gamma_V = \gamma_\mu$,

$$T^V_\mu(k_1) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma_\mu \frac{1}{P'(\not k_1,m)} \right\}, \tag{106}$$

which, after the evaluation of Dirac traces involved, can be written as

$$T^V_\mu(k_1) = 4\{I_{1\mu}(k_1) + k_{1\mu}[I_1(k_1)]\}, \tag{107}$$

where we have used the definitions contained in (1). From the above equation we note that only two divergent integrals have appeared, having cubic and quadratic degrees of divergence, which have been considered in Sect. 5. By using the results (75) and (77) we get

$$T^V_\mu(k_1) = 4\left\{ -k_1^\beta [\nabla_{\beta\mu}] - \frac{1}{3} k_1^\beta k_1^\alpha k_1^\nu [\Box_{\alpha\beta\mu\nu}] + \frac{1}{3} k_1^2 k_1^\nu [\Delta_{\nu\mu}] + \frac{2}{3} k_{1\mu} k_{1\alpha} k_{1\beta} \left[\Delta^{\alpha\beta}\right] \right\}. \tag{108}$$

The amplitude, as promised, has been written in terms of the objects belonging to our systematization. Let us now consider an example of two-point functions.

6.2 The vector–vector two-point function

If one wants to consider a representative Green's function of the perturbative calculation, concerning the consistency in the manipulations and calculations involving divergent Feynman integrals, certainly there is no better one than the vector–vector two-point function related to the QED vacuum polarization tensor, which is given by

$$T^{VV}_{\mu\nu}(k_1,k_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma_\mu \frac{1}{P'(\not k_1,m)} \gamma_\nu \frac{1}{P'(\not k_2,m)} \right\}. \tag{109}$$

After the traces evaluation we get

$$\begin{aligned} T^{VV}_{\mu\nu}(k_1,k_2) = 4\{&2[I_{2\mu\nu}(k_1,k_2)] + P_\nu[I_{2\mu}(k_1,k_2)] \\ &+ P_\nu[I_{2\mu}(k_1,k_2)] \\ &+ (k_{2\mu}k_{1\nu} + k_{1\mu}k_{2\nu})[I_2(k_1,k_2)]\} \\ &+ 4g_{\mu\nu}\left[T^{PP}(k_1,k_2)\right], \end{aligned} \tag{110}$$

where

$$T^{PP}(k_1,k_2) = -2\left\{I_1(k_1) + I_1(k_2) - p^2[I_2(k_1,k_2)]\right\}.$$

Here we have identified the term proportional to $g_{\mu\nu}$, after the traces for the VV two-point function are taken, with that corresponding to the PP amplitude. Such type of decomposition is always possible and we will make use of it in future calculations as a part of our systematization.

Next, in order to complete the calculation we substitute the results obtained in Sect. 5 for the involved Feynman integrals, (77), (81), (85), and (86). The result can be put in the simple form

$$\begin{aligned} T^{VV}_{\mu\nu}(k_1,k_2) &= \frac{4}{3}\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \\ &\times \left\{ I_{\log}(m^2) - \mathrm{i}(4\pi)^{-2}\left[\frac{1}{3} + \frac{2m^2+p^2}{p^2} Z_0\left(p^2;m^2\right)\right]\right\} \\ &+ A_{\mu\nu}(k_1,k_2), \end{aligned} \tag{111}$$

where we have defined

$$\begin{aligned} A_{\mu\nu}(k_1,k_2) &= 4[\nabla_{\mu\nu}] + p^\alpha p^\beta \left\{ \frac{1}{3}[\Box_{\alpha\beta\mu\nu}] + \frac{1}{3} g_{\alpha\nu}[\Delta_{\mu\beta}] \right. \\ &\left. + g_{\alpha\mu}[\Delta_{\beta\nu}] - g_{\mu\nu}[\Delta_{\alpha\beta}] - \frac{2}{3} g_{\alpha\beta}[\Delta_{\mu\nu}] \right\} \\ &+ \left(p^\alpha P^\beta - P^\alpha p^\beta\right) \\ &\times \left\{ \frac{1}{3}[\Box_{\alpha\beta\mu\nu}] + \frac{1}{3} g_{\nu\alpha}[\Delta_{\mu\beta}] + \frac{1}{3} g_{\alpha\mu}[\Delta_{\beta\nu}] \right\} \\ &+ P^\alpha P^\beta \{[\Box_{\alpha\beta\mu\nu}] - g_{\mu\beta}[\Delta_{\nu\alpha}] - g_{\alpha\mu}[\Delta_{\beta\nu}] \\ &- 3g_{\mu\nu}[\Delta_{\alpha\beta}]\}. \end{aligned} \tag{112}$$

Let us now consider a three-point function.

6.3 The triple vector three-point function

As an example of calculation of a Green's function of the perturbative calculations having three fermionic propagators, we consider the triple vector three-point function, given by

$$T^{VVV}_{\beta\nu\alpha}(k_1,k_2,k_3) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma_\beta \frac{1}{P'(\not k_1,m)} \gamma_\nu \frac{1}{P'(\not k_2,m)} \gamma_\alpha \right.$$

$$\times \frac{1}{P'(\not k_3, m)}\Bigg\}, \tag{113}$$

which can be conveniently written as

$$T^{VVV}_{\beta\nu\alpha}(k_1,k_2,k_3) = 4\left[T_{\beta\nu\alpha}\right] + g_{\nu\alpha}\left[T^{VPP}_{\beta}(k_1,k_2,k_3)\right] + g_{\beta\nu}\left[T^{PPV}_{\alpha}(k_1,k_2,k_3)\right] + g_{\beta\alpha}\left[T^{PVP}_{\nu}(k_1,k_2,k_3)\right]. \tag{114}$$

Again the VPP structure appearing must be understood as a compact representation for the terms involved in the traces. The introduced tensor $T_{\beta\nu\alpha}$, on the other hand, is given by

$$\begin{aligned}
T_{\beta\nu\alpha} &= 16\left[I_{3\beta\nu\alpha}(k_1,k_2,k_3)\right] \\
&+8(k_2+k_3)_\alpha\left[I_{3\beta\nu}(k_1,k_2,k_3)\right] \\
&+8(k_1+k_2)_\nu\left[I_{3\beta\alpha}(k_1,k_2,k_3)\right] \\
&+8(k_1+k_3)_\beta\left[I_{3\nu\alpha}(k_1,k_2,k_3)\right] \\
&+4\left[k_{1\alpha}(k_2-k_3)_\nu + k_{1\nu}(k_2+k_3)_\alpha\right. \\
&\left.+(k_{3\alpha}k_{2\nu}+k_{2\alpha}k_{3\nu})\right]\left[I_{3\beta}(k_1,k_2,k_3)\right] \\
&+4\Big[k_{1\beta}(k_2+k_3)_\alpha - k_{1\alpha}(k_2-k_3)_\beta \\
&+k_{2\alpha}k_{3\beta}+k_{3\alpha}k_{2\beta}\Big]\left[I_{3\nu}(k_1,k_2,k_3)\right] \\
&+4\Big[k_{1\beta}(k_2+k_3)_\nu + k_{1\nu}(k_2+k_3)_\beta \\
&+k_{2\nu}k_{3\beta}-k_{3\nu}k_{2\beta}\Big]\left[I_{3\alpha}(k_1,k_2,k_3)\right] \\
&+4\left[k_{1\beta}(k_{2\nu}k_{3\alpha}+k_{3\nu}k_{2\alpha})\right. \\
&+k_{1\nu}(k_{2\alpha}k_{3\beta}+k_{2\beta}k_{3\alpha}) \\
&\left.+k_{1\alpha}(k_{2\nu}k_{3\beta}-k_{3\nu}k_{2\beta})\right]\left[I_3(k_1,k_2,k_3)\right].
\end{aligned} \tag{115}$$

In order to complete the calculation the substitution of the results for the Feynman integrals appearing in the expression above becomes necessary only. By using the results (92), (91), (88), and (87), we write the tensor $T_{\beta\nu\alpha}$ as

$$\begin{aligned}
T_{\beta\nu\alpha} &= T'_{\beta\nu\alpha} + \{g_{\alpha\nu}(q_\beta+p_\beta)+g_{\alpha\beta}(p_\nu-2q_\nu) \\
&+g_{\beta\nu}(q_\alpha-2p_\alpha)\}\left[\frac{2}{3}I_{\log}(m^2)\right] \\
&+4\{g_{\beta\nu}q_\alpha+g_{\alpha\beta}p_\nu+g_{\alpha\nu}(q_\beta+p_\beta)\}(-\eta_{00}) \\
&+4(g_{\alpha\beta}p_\nu+g_{\nu\beta}p_\alpha+g_{\alpha\nu}p_\beta)(2\eta_{01}) \\
&+4(g_{\alpha\beta}q_\nu+g_{\nu\beta}q_\alpha+g_{\alpha\nu}q_\beta)(2\eta_{10}) \\
&+4p_\beta p_\nu p_\alpha(4\xi_{02}-4\xi_{03}) \\
&+4q_\beta q_\alpha q_\nu(4\xi_{20}-4\xi_{30}) \\
&+4p_\beta p_\alpha q_\nu(2\xi_{11}-4\xi_{12}) \\
&+4q_\beta p_\alpha q_\nu(2\xi_{11}-4\xi_{21}) \\
&+4p_\beta q_\alpha q_\nu(2\xi_{20}+2\xi_{11}-4\xi_{21}-2\xi_{10}) \\
&+4q_\beta p_\alpha p_\nu(2\xi_{02}+2\xi_{11}-4\xi_{12}-2\xi_{01}) \\
&+4p_\beta p_\nu q_\alpha(4\xi_{11}+2\xi_{02}-4\xi_{12}-2\xi_{01}) \\
&+4q_\beta q_\alpha p_\nu(4\xi_{11}+2\xi_{20}-4\xi_{21}-2\xi_{10}),
\end{aligned} \tag{116}$$

where

$$\begin{aligned}
T'_{\beta\nu\alpha} &= -\frac{1}{3}(k_1^\eta+k_2^\eta+k_3^\eta)(\Box_{\eta\beta\nu\alpha}) \\
&+\frac{1}{6}(q_\beta+p_\beta)(\triangle_{\alpha\nu}) \\
&-\frac{1}{6}(2q_\nu-p_\nu)(\triangle_{\alpha\beta})-\frac{1}{6}(2p_\alpha-q_\alpha)(\triangle_{\beta\nu}).
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
T^{VPP}_{\beta}(p,q) &= -2(p+q)_\beta\left[I_{\log}(m^2)\right] \\
&+2\mathrm{i}(4\pi)^{-2}q_\beta\Big\{Z_0(q^2;m^2)+p^2\xi_{00}(p,q) \\
&+\left((p-q)^2-p^2-q^2\right)\xi_{10}(p,q)\Big\} \\
&-2\mathrm{i}(4\pi)^{-2}p_\beta\Big\{Z_0(p^2;m^2)+q^2\xi_{00}(p,q) \\
&+\left((p-q)^2-p^2-q^2\right)\xi_{01}(p,q)\Big\},
\end{aligned} \tag{117}$$

$$\begin{aligned}
T^{PPV}_{\nu} &= 2(2q_\nu-p_\nu)\left[I_{\log}(m^2)\right] \\
&-2\mathrm{i}(4\pi)^{-2}q_\nu\Big\{Z_0\left((p-q)^2;m^2\right)+Z_0(q^2;m^2) \\
&+p^2(\xi_{00})+\left(q^2-p^2+(p-q)^2\right)(\xi_{10})\Big\} \\
&+2\mathrm{i}(4\pi)^{-2}p_\nu\Big\{Z_0(p^2;m^2)-q^2(\xi_{00}) \\
&+\left(p^2-q^2-(p-q)^2\right)(\xi_{01})\Big\},
\end{aligned} \tag{118}$$

$$\begin{aligned}
T^{PVP}_{\alpha} &= 2(2p_\alpha-q_\alpha)\left[I_{\log}(m^2)\right] \\
&+2\mathrm{i}(4\pi)^{-2}q_\alpha\Big\{Z_0\left((p-q)^2;m^2\right)+p^2(\xi_{00}) \\
&+\left(q^2-p^2-(p-q)^2\right)(\xi_{10})\Big\} \\
&-2\mathrm{i}(4\pi)^{-2}p_\alpha\Big\{Z_0\left((p-q)^2;m^2\right)+Z_0(p^2;m^2) \\
&+q^2(\xi_{00})+\left(p^2-q^2+(p-q)^2\right)(\xi_{01})\Big\},
\end{aligned} \tag{119}$$

which completes the calculation of the VVV amplitude. Finally, we consider the evaluation of the four-point function.

6.4 The four-vector four-point function

As an example of a calculation of a Green's function of the perturbative calculations having four fermionic propagators, we consider the four-vector four-point function, given by

$$T^{VVVV}_{\mu\nu\alpha\beta}(k_1,k_2,k_3,k_4) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{Tr}\left\{\gamma_\mu \frac{1}{P'(\not k_1,m)}\gamma_\nu \frac{1}{P'(\not k_2,m)}\gamma_\alpha \times \frac{1}{P'(\not k_3,m)}\gamma_\beta \frac{1}{P'(\not k_4,m)}\right\}. \quad (120)$$

After performing the Dirac traces, and some algebraic effort, it is convenient to identify the decomposition

$$\begin{aligned} T^{VVVV}_{\mu\nu\alpha\beta}(p,q,r) &= T_{\mu\nu\alpha\beta}(p,q,r) + g_{\alpha\beta}\left[T^{VVPP}_{\mu\nu}(p,q,r)\right] \\ &+ g_{\mu\nu}\left[T^{PPVV}_{\alpha\beta}(p,q,r)\right] + g_{\nu\alpha}\left[T^{VPPV}_{\mu\beta}(p,q,r)\right] \\ &+ g_{\nu\beta}\left[T^{VPVP}_{\mu\alpha}(p,q,r)\right] + g_{\mu\alpha}\left[T^{PVPV}_{\nu\beta}(p,q,r)\right] \\ &+ g_{\mu\beta}\left[T^{PVVP}_{\nu\alpha}(p,q,r)\right] \qquad (121) \\ &- (g_{\mu\nu}g_{\alpha\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta})\left[T^{PPPP}(p,q,r)\right], \end{aligned}$$

where we have identified the terms in the traces operation proportional to tensor metrics with those corresponding to the four-point function amplitudes $VVPP$ and $PPPP$. In addition we have introduced the tensor $T_{\mu\nu\alpha\beta}$

$$\begin{aligned} T_{\mu\nu\alpha\beta} &= 8\left[I_{4\mu\nu\alpha\beta}(p,q,r)\right] + 4p_\nu\left[I_{4\mu\alpha\beta}(p,q,r)\right] \\ &+ 4r_\mu\left[I_{4\nu\alpha\beta}(p,q,r)\right] \\ &+ 4(r+q)_\beta\left[I_{4\mu\nu\alpha}(p,q,r)\right] \\ &+ 4(p+q)_\alpha\left[I_{4\mu\nu\beta}(p,q,r)\right] \\ &+ 2\Big[r_\beta(p+q)_\alpha - r_\alpha(p-q)_\beta + q_\beta p_\alpha + q_\alpha p_\beta\Big] \\ &\times\left[I_{4\mu\nu}(p,q,r)\right] \\ &+ 2(p_\nu r_\mu - p_\mu r_\nu)\left[I_{4\alpha\beta}(p,q,r)\right] + 2(p_\nu q_\alpha + p_\alpha q_\nu) \\ &\times\left[I_{4\mu\beta}(p,q,r)\right] \\ &+ 2\left[r_\beta p_\nu + r_\nu p_\beta + p_\nu q_\beta - q_\nu p_\beta\right]\left[I_{4\mu\alpha}(p,q,r)\right] \\ &+ 2(r_\beta q_\mu + r_\mu q_\beta)\left[I_{4\nu\alpha}(p,q,r)\right] \\ &+ 2\Big[r_\alpha(p-q)_\mu + r_\mu(p+q)_\alpha\Big]\left[I_{4\nu\beta}(p,q,r)\right] \\ &+ \left[p_\nu(q_\alpha r_\beta + r_\alpha q_\beta) + r_\nu(q_\beta p_\alpha + q_\alpha p_\beta)\right. \\ &\left.+ q_\nu(r_\beta p_\alpha - r_\alpha p_\beta)\right]\left[I_{4\mu}(p,q,r)\right] \\ &+ \left[p_\mu(r_\beta q_\alpha + r_\alpha q_\beta) + r_\mu(q_\beta p_\alpha + q_\alpha p_\beta)\right. \\ &\left.+ q_\mu(r_\beta p_\alpha - r_\alpha p_\beta)\right]\left[I_{4\nu}(p,q,r)\right] \\ &+ \left[r_\mu(q_\beta p_\nu - q_\nu p_\beta) - p_\mu(r_\nu q_\beta + r_\beta q_\nu)\right. \\ &\left.+ q_\mu(r_\beta p_\nu + r_\nu p_\beta)\right]\left[I_{4\alpha}(p,q,r)\right] \\ &+ \left[r_\mu(p_\nu q_\alpha + p_\alpha q_\nu) - q_\mu(p_\nu r_\alpha + p_\alpha r_\nu)\right. \\ &\left.+ p_\mu(r_\alpha q_\nu - r_\nu q_\alpha)\right] I\left[_{4\beta}(p,q,r)\right]. \qquad (122) \end{aligned}$$

From the results of the integrals listed in Sect. 5 it is an easy task, but a tedious one, to obtain expressions for the above amplitudes. We do not present explicit expressions for these calculations because their length is prohibitive. As an example, by using the results (94), (87), and (81) we can write out the four-point $PPPP$ amplitude as

$$\begin{aligned} T^{PPPP} &= 4\left[I_{\log}\left(m^2\right)\right] \\ &-2\left[Z_0\left(u^2,m^2\right) + Z_0\left(q^2,m^2\right)\right] - 2(r\cdot p)\,\xi_{00}(r,p) \\ &-2\left[q^2 + (r\cdot p) - (r\cdot q) - (p\cdot q)\right]\xi_{00}(u,t) \\ &-2\left[r^2 - (r\cdot q)\right]\xi_{00}(r,q) \\ &-2\left[p^2 - (p\cdot q)\right]\xi_{00}(p,q) \\ &+\left(r^2s^2 - q^2u^2 + p^2t^2\right)\zeta_{000}(p,q,r). \qquad (123) \end{aligned}$$

Our main purpose have been, at this point, fulfilled which is to show how the proposed systematization works in the calculation of physical amplitudes. However, another important aspect involved in perturbative calculations can be also considered which, within the context of our procedure, became very simple and transparent: that is the verification of relations among the Green's functions and, consequently, of the associated Ward identities. We perform such a task in the next section.

7 Relations among Green's functions

The procedure which we have described to systematize the perturbative calculations in QFT is undoubtedly very general once no particular assumptions have been made in the intermediary steps of the calculations performed when the involved integrals are divergent quantities. However, it is easy to note that in the results obtained for the calculated amplitudes, there are many kinds of arbitrariness which only can be removed after the adoption of a certain set of choices. Usually such choices are made automatically when a regularization or equivalent philosophy is adopted. The aim of the procedure adopted in the present discussions is the preservation of the general character as much as possible. Therefore, the relevant question at this moment is the following: do the manipulations made preserve the relations among Green's functions which can be stated at the level of the integrand? The answer for this question is very relevant because the preservation of the symmetry relations pertinent to the evaluated amplitudes are intimately connected to this aspect. Having this in mind, in the present section, we consider the verification of the relations existing among the considered amplitudes without assuming specific choices for the intrinsic arbitrariness involved in the calculations, i.e., in spite of having arbitrary pieces we will verify if the relations are preserved by the manipulations made.

We start by considering the VV two-point function. It is immediate to note the identity

$$p^\mu\left\{\gamma_\nu \frac{1}{(\not k + \not k_1) - m}\gamma_\mu \frac{1}{(\not k + \not k_2) - m}\right\}$$

$$= \gamma_\nu \frac{1}{\not k + \not k_1 - m} - \gamma_\nu \frac{1}{\not k + \not k_2 - m}, \quad (124)$$

which, after taking the Dirac traces and integrating over the k momentum in both sides, allows us to identify a relation between two physical amplitudes, which is

$$p^\mu T^{VV}_{\mu\nu}(k_1,k_2) = T^V_\nu(k_1) - T^V_\nu(k_2). \quad (125)$$

In a similar way we can also state that

$$p^\nu T^{VV}_{\mu\nu}(k_1,k_2) = T^V_\mu(k_1) - T^V_\mu(k_2). \quad (126)$$

The above relation tells us that if we explicitly calculate the $T^{VV}_{\mu\nu}(k_1,k_2)$ and after this contract the obtained expression with the external momentum, we must identify, in the so obtained result, the difference between two one-point vector functions with internal lines carrying momentum k_1 and k_2 (which are arbitrary). We have evaluated explicitly both amplitudes involved in the above identity so that we can verify if the relation is preserved by the obtained results. Contracting the expression (111) with the momentum p_μ we note that the finite part is removed and we see that

$$p^\mu T^{VV}_{\mu\nu}(k_1,k_2) = p^\mu A_{\mu\nu}, \quad (127)$$

where $A_{\mu\nu}$ was defined in (112). Next it is simple to reorganize the remaining terms to see that

$$p^\mu A_{\mu\nu} = T^V_\nu(k_1) - T^V_\nu(k_2), \quad (128)$$

$$p^\nu A_{\mu\nu} = T^V_\mu(k_1) - T^V_\mu(k_2), \quad (129)$$

which implies that the relations between the VV two-point function with the one-point vector function are preserved by the manipulations made. Note that the relations are preserved without any assumption about the undefined quantities so that it can be preserved by any reasonable regularization method. The constraints imposed by the symmetry implications over the amplitudes we consider in the next section.

Now we consider the triple vector triangle amplitude and their relations between the VV two-point functions. First we note the identity

$$p^\nu \left\{ \gamma_\alpha \frac{1}{(\not k + \not k_1) - m} \gamma_\nu \frac{1}{(\not k + \not k_2) - m} \gamma_\beta \times \frac{1}{(\not k + \not k_3) - m} \right\}$$
$$= \gamma_\nu \frac{1}{\not k + \not k_1 - m} \gamma_\beta \frac{1}{(\not k + \not k_3) - m} - \gamma_\nu \frac{1}{\not k + \not k_2 - m} \gamma_\beta \frac{1}{(\not k + \not k_3) - m}. \quad (130)$$

Again, after taking the traces and integrating in the momentum k a relation between two amplitudes of the perturbative calculations, the triple vector triangle and the VV two-point function can be identified, which can be written as

$$p^\nu T^{VVV}_{\beta\alpha\nu}(k_1,k_2,k_3) = T^{VV}_{\beta\alpha}(k_3,k_1) - T^{VV}_{\beta\alpha}(k_3,k_2). \quad (131)$$

Similar relations can be stated taking the contraction with the remaining external momenta. They are

$$(q-p)^\alpha T^{VVV}_{\beta\alpha\nu}(k_1,k_2,k_3) = T^{VV}_{\beta\nu}(k_1,k_2) - T^{VV}_{\beta\nu}(k_3,k_1), \quad (132)$$

$$q^\beta T^{VVV}_{\beta\alpha\nu}(k_1,k_2,k_3) = T^{VV}_{\alpha\nu}(k_1,k_2) - T^{VV}_{\alpha\nu}(k_3,k_2) \quad (133)$$

In order to verify if the obtained expressions for the involved amplitudes are compatible with the identities stated above, we take (114) and contract with the external momenta of the vertices. Let us start by the contraction with the momentum p_ν, which is the incoming momentum according to our convention. The contraction gives

$$\begin{aligned}
p^\nu T^{VVV}_{\beta\alpha\nu} &= \{p_\alpha(q_\beta+p_\beta) + p_\beta(q_\alpha - 2p_\alpha) \\
&\quad + g_{\alpha\beta}[p^2 - 2(p\cdot q)]\} \left[\frac{2}{3} I_{\log}(m^2)\right] \\
&\quad +4\{p_\beta q_\alpha + g_{\alpha\beta}p^2 + p_\alpha(q_\beta+p_\beta)\}(-\eta_{00}) \\
&\quad +4(g_{\alpha\beta}p^2 + p_\beta p_\alpha + p_\alpha p_\beta)(2\eta_{10}) \\
&\quad +4(g_{\alpha\beta}(p\cdot q) + p_\beta q_\alpha + p_\alpha q_\beta)(2\eta_{01}) \\
&\quad +4p_\beta p_\alpha\{p^2(-4\xi_{30}) + (p\cdot q)(-4\xi_{21})\} \\
&\quad +4q_\beta q_\alpha\{p^2(-4\xi_{12}) + (p\cdot q)(-4\xi_{03})\} \\
&\quad +4q_\beta p_\alpha\{p^2(-4\xi_{21}) + (p\cdot q)(-4\xi_{12})\} \\
&\quad +4p_\beta q_\alpha\{p^2(-4\xi_{21}) + (p\cdot q)(-4\xi_{12})\} \\
&\quad +4p_\beta p_\alpha\{p^2(4\xi_{20}) + (p\cdot q)(2\xi_{11})\} \\
&\quad +4q_\beta q_\alpha(p\cdot q)(4\xi_{02}) + 4q_\beta p_\alpha(p\cdot q)(2\xi_{11}) \\
&\quad +4p_\beta q_\alpha(p\cdot q)(2\xi_{02} + 2\xi_{11} - 2\xi_{01}) \\
&\quad +4q_\beta p_\alpha p^2(2\xi_{20} + 2\xi_{11} - 2\xi_{10}) \\
&\quad +4p_\beta q_\alpha p^2(4\xi_{11} + 2\xi_{20} - 2\xi_{10}) \\
&\quad +4q_\beta q_\alpha p^2(4\xi_{11} + 2\xi_{02} - 2\xi_{01}) \\
&\quad +p_\alpha[T^{VPP}_\beta] + g_{\alpha\beta}[p^\nu T^{PPV}_\nu] + p_\beta[T^{PVP}_\alpha] \\
&\quad +A_{\mu\nu}(k_1,k_3) - A_{\mu\nu}(k_3,k_2),
\end{aligned} \quad (134)$$

where we have conveniently completed the divergent terms in order to identify the $A_{\mu\nu}$ terms. The next step is the use of the properties (38), (39), and (40) in order to eliminate the ξ_{nm} functions having $n+m=3$ in favor of those having $n+m=2$. Then we get

$$\begin{aligned}
p^\nu T^{VVV}_{\beta\alpha\nu} &= \{p_\alpha(q_\beta+p_\beta) + p_\beta(q_\alpha - 2p_\alpha) \\
&\quad + g_{\alpha\beta}[p^2 - 2(p\cdot q)]\} \left[\frac{2}{3} I_{\log}(m^2)\right] \\
&\quad +4p_\beta p_\alpha\left\{2Z_2\left((p-q)^2; m^2\right)\right\}
\end{aligned}$$

$$+4q_\alpha q_\beta\left\{2Z_2\left((p-q)^2;m^2\right)-2Z_2\left(q^2;m^2\right)\right\}$$
$$+4q_\beta p_\alpha\left\{-2Z_2\left((p-q)^2;m^2\right)+Z_0\left((p-q)^2;m^2\right)\right\}$$
$$+4p_\beta q_\alpha\left\{-2Z_2\left((p-q)^2;m^2\right)+Z_0\left((p-q)^2;m^2\right)\right\}$$
$$+4\left\{p_\beta q_\alpha+g_{\alpha\beta}p^2+p_\alpha\left(q_\beta+p_\beta\right)\right\}\left(-\eta_{00}\right)$$
$$+4g_{\alpha\beta}\left\{p^2\left(2\eta_{10}\right)+(p\cdot q)\left(2\eta_{01}\right)\right\}$$
$$+4p_\beta p_\alpha\left\{p^2\left(2\xi_{20}\right)+(p\cdot q)\left(2\xi_{11}\right)\right\}$$
$$+4q_\beta q_\alpha\left\{p^2\left(4\xi_{11}\right)+(p\cdot q)\left(4\xi_{02}\right)\right\}$$
$$+4q_\beta p_\alpha\left\{p^2\left(2\xi_{20}\right)+(p\cdot q)\left(2\xi_{11}\right)\right\}$$
$$+4p_\beta q_\alpha\left\{p^2\left(2\xi_{11}\right)+(p\cdot q)\left(2\xi_{02}\right)\right\}$$
$$+4p_\beta q_\alpha\left\{p^2\left(2\xi_{20}\right)+(p\cdot q)\left(2\xi_{11}\right)\right\}$$
$$+4p_\beta q_\alpha(p\cdot q)\left(-2\xi_{01}\right)+4q_\beta p_\alpha p^2\left(-2\xi_{10}\right)$$
$$+4p_\beta q_\alpha p^2\left(-2\xi_{10}\right)+4q_\beta q_\alpha p^2\left(-2\xi_{01}\right)$$
$$+p_\alpha\left[T_\beta^{VPP}\right]+g_{\alpha\beta}\left[p^\nu T_\nu^{PPV}\right]+p_\beta\left[T_\alpha^{PVP}\right]$$
$$+A_{\mu\nu}\left(k_1,k_3\right)-A_{\mu\nu}\left(k_3,k_2\right). \qquad (135)$$

Given the obtained result, we now use the properties (36), (37) and (41) to eliminate the ξ_{nm} functions having $n+m=2$ in favor of those having $n+m=1$. Furthermore we use the results (117) and (119) to get

$$p^\nu T_{\beta\alpha\nu}^{VVV}$$
$$=\frac{2}{3}\left[g_{\alpha\beta}(p-q)^2+2(p-q)_\alpha(p-q)_\beta\right]\left[I_{\log}\left(m^2\right)\right]$$
$$-\frac{2}{3}\left(g_{\alpha\beta}q^2+2q_\alpha q_\beta\right)\left[I_{\log}\left(m^2\right)\right]$$
$$-2g_{\alpha\beta}(p-q)^2$$
$$\times\left[4Z_2\left((p-q)^2;m^2\right)-Z_0\left((p-q)^2;m^2\right)\right]$$
$$+2g_{\alpha\beta}q^2\left[4Z_2\left(q^2;m^2\right)-Z_0\left(q^2;m^2\right)\right]$$
$$+4(p-q)_\beta(p-q)_\alpha$$
$$\times\left\{2Z_2\left((p-q)^2;m^2\right)-Z_0\left((p-q)^2;m^2\right)\right\}$$
$$-2\left(p_\beta q_\alpha+q_\beta p_\alpha\right)\left\{Z_0\left((p-q)^2;m^2\right)-Z_0\left(q^2;m^2\right)\right\}$$
$$+4q_\beta q_\alpha\left\{-2Z_2\left(q^2;m^2\right)+Z_0\left(q^2;m^2\right)\right\}$$
$$+4q_\beta p_\alpha\left\{p^2\left(-\xi_{10}\right)+(p\cdot q)\left(-\xi_{01}\right)\right\}$$
$$+4p_\beta q_\alpha\left\{p^2\left(-\xi_{10}\right)+(p\cdot q)\left(-\xi_{01}\right)\right\}$$
$$-2p_\alpha q_\beta p^2\left(-\xi_{00}\right)-2p_\beta q_\alpha p^2\left(-\xi_{00}\right)$$
$$+g_{\alpha\beta}\left[T^{PP}\left(k_1,k_3\right)-T^{PP}\left(k_2,k_3\right)\right]$$
$$+A_{\mu\nu}\left(k_1,k_3\right)-A_{\mu\nu}\left(k_3,k_2\right). \qquad (136)$$

In the last step we eliminate the ξ_{nm} functions having $n+m=1$ through the property (35) and note that the expression above may be conveniently reorganized as

$$p^\nu T_{\beta\alpha\nu}^{VVV}$$
$$=\frac{2}{3}\left[g_{\alpha\beta}(p-q)^2+2(p-q)_\alpha(p-q)_\beta\right]\left[I_{\log}\left(m^2\right)\right]$$
$$-4(p-q)_\beta(p-q)_\alpha$$
$$\times\left\{-2Z_2\left((p-q)^2;m^2\right)+Z_0\left((p-q)^2;m^2\right)\right\}$$
$$-4g_{\alpha\beta}(p-q)^2$$
$$\times\left[2Z_2\left((p-q)^2;m^2\right)-\frac{1}{2}Z_0\left((p-q)^2;m^2\right)\right]$$
$$-g_{\alpha\beta}\left[T^{PP}\left(k_2,k_3\right)\right]-A_{\mu\nu}\left(k_3,k_2\right)$$
$$-\frac{2}{3}\left(g_{\alpha\beta}q^2+2q_\alpha q_\beta\right)\left[I_{\log}\left(m^2\right)\right]$$
$$+4q_\beta q_\alpha\left\{-2Z_2\left(q^2;m^2\right)+Z_0\left(q^2;m^2\right)\right\}$$
$$+4g_{\alpha\beta}q^2\left[2Z_2\left(q^2;m^2\right)-\frac{1}{2}Z_0\left(q^2;m^2\right)\right]$$
$$+g_{\alpha\beta}\left[T^{PP}\left(k_1,k_3\right)\right]+A_{\mu\nu}\left(k_1,k_3\right). \qquad (137)$$

Finally, from the expression (111) for the VV two-point function we see that relation (131) is satisfied. It is not difficult to verify the relations (133) and (132) by performing the same sequence of steps.

The procedure used above can also be adopted to state four constraints to the four-vector Green's function. They are

$$r^\mu T_{\mu\nu\alpha\beta}^{VVVV}\left(k_1,k_2,k_3,k_4\right)$$
$$=T_{\nu\alpha\beta}^{VVV}\left(k_1,k_2,k_3\right)-T_{\nu\alpha\beta}^{VVV}\left(k_2,k_3,k_4\right), \qquad (138)$$
$$p^\nu T_{\mu\nu\alpha\beta}^{VVVV}\left(k_1,k_2,k_3,k_4\right)$$
$$=T_{\mu\alpha\beta}^{VVV}\left(k_1,k_3,k_4\right)-T_{\beta\mu\alpha}^{VVV}\left(k_2,k_3,k_4\right), \qquad (139)$$
$$(q-p)^\alpha T_{\mu\nu\alpha\ \beta}^{VVVV}\left(k_1,k_2,k_3,k_4\right)$$
$$=T_{\mu\nu\beta}^{VVV}\left(k_1,k_2,k_4\right)-T_{\mu\nu\ \beta}^{VVV}\left(k_1,k_3,k_4\right), \qquad (140)$$
$$(r-q)^\beta T_{\mu\nu\alpha\ \beta}^{VVVV}\left(k_1,k_2,k_3,k_4\right)$$
$$=T_{\mu\nu\alpha}^{VVV}\left(k_1,k_2,k_3\right)-T_{\mu\nu\alpha}^{VVV}\left(k_1,k_2,k_4\right). \qquad (141)$$

In order to verify if the obtained expression (121) are compatible with the identities stated above, we can follow the same steps we have performed in the one-, two- and three-point vector functions. That is, we first contract the amplitude with the external momentum attached to the respective vertex. The next step is to use the properties (64)–(73) in order to eliminate the ζ_{nml} and ξ_{nml} functions having $n+m+l=4$ in favor of those having $n+m+l=3$. From the obtained result, we can use the properties (58)–(63) to eliminate the ζ_{nml} and ξ_{nml} functions having $n+m+l=3$ in favor of those having $n+m+l=2$ and so on. The divergent parts can be conveniently reorganized in order to allow the identification of the two triple vector three-point functions.

8 Ambiguities and Ward identities

In Sect. 6 we have evaluated, within the systematization proposed, Green's functions which are typical of the perturbative calculations. In particular, all the considered amplitudes appear in the context of QED. In all the evaluated Green's functions, having degrees of divergence higher than the logarithmic one, it is possible to note the presence of terms where the dependence on the internal momenta appear in ambiguous combinations (the summations of them). This is expected since a shift in the integrating momentum generates surfaces terms which implies that different choices for the labels of the internal lines momenta lead to different amplitudes, which characterizes ambiguities. In the case of the vector one-point function $T_\nu^V(k_1)$ all the terms are dependent on k_1, which is arbitrary. For the two-point function $T_{\mu\nu}^{VV}(k_1,k_2)$, we can identify the ambiguous terms

$$\begin{aligned}&\left(T_{\mu\nu}^{VV}\right)_{\text{ambig}}\\&=\left(p^\alpha P^\beta-P^\alpha p^\beta\right)\\&\quad\times\frac{1}{3}\left\{[\Box_{\alpha\beta\mu\nu}]+g_{\nu\alpha}\left[\Delta_{\mu\beta}\right]+g_{\alpha\mu}\left[\Delta_{\beta\nu}\right]\right\}\\&\quad+P^\alpha P^\beta\\&\quad\times\left\{[\Box_{\alpha\beta\mu\nu}]-g_{\mu\beta}\left[\Delta_{\nu\alpha}\right]-g_{\alpha\mu}\left[\Delta_{\beta\nu}\right]-3g_{\mu\nu}\left[\Delta_{\alpha\beta}\right]\right\}.\end{aligned}\tag{142}$$

Finally, in the evaluation of the triple vector three-point function we see that

$$\left(T_{\beta\nu\alpha}^{VVV}\right)_{\text{ambig}}=-\frac{1}{3}\left(k_1^\eta+k_2^\eta+k_3^\eta\right)\left(\Box_{\eta\beta\nu\alpha}\right).\tag{143}$$

Concerning the symmetry relations the situation is similar. The Furry theorem states that every amplitude which has an even number of external vectors and only one species of fermion at the internal lines must vanish identically. This means that the amplitude $T_\nu^V(k_1)$ must be zero as well as the symmetrized final states of the triple vector three-point function, which we define as

$$T_{\beta\nu\alpha}^{V\to VV}=T_{\beta\nu\alpha}^{VVV}\left(k_1,k_2,k_3\right)+T_{\beta\alpha\nu}^{VVV}\left(l_1,l_2,l_3\right)\tag{144}$$

($q=l_2-l_1$ and $p=l_3-l_1$) must vanish too. In the context of gauge symmetries, like QED, it is required that all vector currents are conserved. This means that all the contractions with external momenta must vanish identically. This argument implies that for the vector one-point function

$$\begin{aligned}k_1^\mu T_\mu^V\left(k_1\right)&=-4k_1^\mu\left\{k_1^\beta\left[\nabla_{\beta\mu}\right]+\frac{1}{3}k_1^\alpha k_1^\beta k_1^\nu\left[\Box_{\alpha\beta\mu\nu}\right]\right.\\&\qquad\left.-\frac{1}{3}k_1^2k_1^\nu\left[\Delta_{\nu\mu}\right]-\frac{2}{3}k_{1\mu}k_{1\alpha}k_{1\beta}\left[\Delta^{\alpha\beta}\right]\right\},\\&=0,\end{aligned}\tag{145}$$

for the VV two-point function,

$$\begin{aligned}&p^\mu T_{\nu\mu}^{VV}\left(k_1,k_2\right)\\&=4[\nabla_{\mu\nu}]+p^\alpha p^\beta\left\{\frac{1}{3}\left[\Box_{\alpha\beta\mu\nu}\right]+\frac{1}{3}g_{\alpha\nu}\left[\Delta_{\mu\beta}\right]\right.\\&\qquad\left.+g_{\alpha\mu}\left[\Delta_{\beta\nu}\right]-g_{\mu\nu}\left[\Delta_{\alpha\beta}\right]-\frac{2}{3}g_{\alpha\beta}\left[\Delta_{\mu\nu}\right]\right\}\\&\quad+\frac{1}{3}\left(p^\alpha P^\beta-P^\alpha p^\beta\right)\\&\quad\times\left\{[\Box_{\alpha\beta\mu\nu}]+g_{\nu\alpha}\left[\Delta_{\mu\beta}\right]+g_{\alpha\mu}\left[\Delta_{\beta\nu}\right]\right\}\\&\quad+P^\alpha P^\beta\\&\quad\times\left\{[\Box_{\alpha\beta\mu\nu}]-g_{\mu\beta}\left[\Delta_{\nu\alpha}\right]-g_{\alpha\mu}\left[\Delta_{\beta\nu}\right]-3g_{\mu\nu}\left[\Delta_{\alpha\beta}\right]\right\},\\&=T_\nu^V\left(k_2,m\right)-T_\nu^V\left(k_1,m\right)=0,\end{aligned}\tag{146}$$

and for the triple vector three-point function,

$$\begin{aligned}&(q-p)^\alpha\,T_{\alpha\nu\beta}^{V\to VV}\\&=T_{\beta\nu}^{VV}\left(k_1,k_2\right)-T_{\beta\nu}^{VV}\left(k_3,k_1\right)\\&\quad+T_{\nu\beta}^{VV}\left(l_1,l_2\right)-T_{\nu\beta}^{VV}\left(l_3,l_1\right)\\&=0.\end{aligned}\tag{147}$$

Finally, for the symmetrized $VVVV$ four-point function, we must have, for example, $r^\mu T_{\mu\nu\alpha\beta}^{V\to VVV}=0$, which implies a condition involving triple vector three-point functions. Similar constraints can be stated for the remaining contractions. All the above conditions for the ambiguities elimination and symmetry preservation can be used to select a class of consistent regularizations: that satisfying what we denominated consistency relations, which are

$$\nabla_{\mu\nu}^{\text{reg}}=\Box_{\alpha\beta\mu\nu}^{\text{reg}}=\Delta_{\mu\beta}^{\text{reg}}=0.$$

Extensive discussions about this aspect can be found in [2, 24].

9 Generalizations of the finite functions and their relationship

Through the proposed method to manipulate and calculate divergent integrals, in the above sections we have been learning how to systematize the finite parts of the one-, two-, three-, and four-point integrals which are present in the relevant amplitudes belonging to fundamental theories. In particular, we saw that the finite parts of the integrals can be organized in two sets of functions: the first set involving the ln function and the second one not. Although it is not the main purpose of this work, in this section we show how to extend those results to an arbitrary number of points and at the same time to unify the notation.

Let us start by defining the first set of functions as

$$\begin{aligned}&\eta_{i_1,\ldots,i_k}^{(n)}\left(p_1,\ldots,p_k\right)\\&=\frac{1}{n!}\int_0^1\mathrm{d}x_1\ldots\mathrm{d}x_k\left[x_1^{i_1}\ldots x_k^{i_k}\right]\left[Q\left(p_1,x_1;\ldots;p_k,x_k\right)\right]^n\\&\quad\times\left\{\ln\frac{Q\left(p_1,x_1;\ldots;p_k,x_k\right)}{-m^2}-\sum_{k'=1}^{n}\frac{1}{k'}\right\},\end{aligned}\tag{148}$$

and the second one, which is related to the $\eta^{(0)}_{i_1,\ldots,i_k}$ functions, through a derivative, as follows:

$$\xi^{(n)}_{i_1,\ldots,i_k} = \frac{(-1)^n}{n!}\frac{\partial^{n+1}\eta^{(0)}_{i_1,\ldots,i_k}}{\partial Q^{n+1}} = \int_0^1 \mathrm{d}x_1\ldots\mathrm{d}x_k \frac{x_1^{i_1}\ldots x_k^{i_k}}{\left[Q\left(p_1,x_1;\ldots;p_k,x_k\right)\right]^{n+1}}, \tag{149}$$

where $k=1,2,3,\ldots$, $n=0,1,2,\ldots$, and

$$Q\left(p_1,x_1;\ldots;p_k,x_k\right) = \sum_{i=1}^{k} p_i^2 x_i\left(1-x_i\right) - 2\sum_{i=1}^{k}\sum_{j>i}^{k}\left(p_i\cdot p_j\right)x_i x_j - m^2. \tag{150}$$

We recognize that (148) is the generalization of definitions (12), (23) and (45) and (149) is the generalization of (22), (44) and (43). These two classes of functions are sufficient to systematize any result (finite part) obtained for Feynman integrals for an arbitrary number of points using the proposed approach. As an additional comment we note that the $\eta^{(n)}_{i_1,\ldots,i_k}$ functions appear only in divergent integrals, having a superficial degree of divergence given by the uppercase index n, while the $\xi^{(n)}_{i_1,\ldots,i_k}$ appear in both, finite and divergent.

It is a common task after evaluation of an amplitude to verify if the symmetry content of the theory is still present which, in general, is not trivial because of the divergences. We have explicitly evaluated the one-, two-, three-, and four-point vector amplitudes and verified their Ward identities. We saw that the verification of the Ward identities is greatly simplified when convenient relations or identities characteristic of $\xi^{(n)}_{i_1,\ldots,i_k}$ and $\eta^{(n)}_{i_1,\ldots,i_k}$ functions are identified. More specifically, we are referring to the relations (35)–(73). Then, it is relevant to study their generalizations.

First, as a matter of notation we define

$$\eta \equiv \eta\left(p_1,\ldots,p_k\right),\quad \eta' \equiv \eta\left(p_2,\ldots,p_k\right),\quad \eta'' \equiv \eta\left(p_1-p_2,\ldots,p_1-p_k\right).$$

In order to achieve the desired results we can integrate (148) by parts. By simple algebraic manipulations we can write

$$\begin{aligned} p_1^2\eta^{(n)}_{i_1,\ldots,i_k} &= -\frac{1}{2n!}\int_0^1 \mathrm{d}x_1\ldots\mathrm{d}x_k\left[x_1^{i_1-1}x_2^{i_2}\ldots x_k^{i_k}\right]\left(\frac{\partial Q}{\partial x_1}\right) \times Q^n\left\{\ln\frac{Q}{-m^2}-\sum_{k'=1}^{n}\frac{1}{k'}\right\} \\ &\quad -\sum_{j=2}^{k}\frac{(p_1\cdot p_j)}{n!}\int_0^1 \mathrm{d}x_1\ldots\mathrm{d}x_k \times\left[x_1^{i_1-1}\ldots x_{j-1}^{i_{j-1}}x_j^{i_j+1}x_{j+1}^{i_{j+1}}\ldots\right] \times Q^n\left\{\ln\frac{Q}{-m^2}-\sum_{k'=1}^{n}\frac{1}{k'}\right\} \\ &\quad +\frac{p_1^2}{2n!}\int_0^1 \mathrm{d}x_1\ldots\mathrm{d}x_k\left[x_1^{i_1-1}x_2^{i_2}\ldots x_k^{i_k}\right] \times Q^n\left\{\ln\frac{Q}{-m^2}-\sum_{k'=1}^{n}\frac{1}{k'}\right\}, \end{aligned} \tag{151}$$

or, in a more convenient form,

$$\begin{aligned} &p_1^2\eta^{(n)}_{i_1,\ldots,i_k} + \left(p_1\cdot p_2\right)\eta^{(n)}_{i_1-1,i_2+1,i_3,\ldots,i_k} + \ldots + \left(p_1\cdot p_k\right)\eta^{(n)}_{i_1-1,i_2,\ldots,i_{k-1},i_k+1} \\ &= -\frac{1}{2n!}\int_0^1 \mathrm{d}x_1\ldots\mathrm{d}x_k\left[x_1^{i_1-1}x_2^{i_2}\ldots x_k^{i_k}\right]\left(\frac{\partial Q}{\partial x_1}\right) \times Q^n\left\{\ln\left(\frac{Q}{-m^2}\right)-\sum_{k'=1}^{n}\frac{1}{k'}\right\} + \frac{1}{2}p_1^2\eta^{(n)}_{i_1-1,i_2,\ldots,i_k}. \end{aligned} \tag{152}$$

Here we have to keep in mind that the conditions $i_1+i_2+\ldots+i_k=1,2,3,\ldots$ with $i_1>0$ must be satisfied. Next, we evaluate the first term on the right hand side, which we call I, using integration by parts:

$$\begin{aligned} I &= \frac{1}{(n+1)!}\int_0^1 \mathrm{d}x_1\ldots\mathrm{d}x_k\frac{\partial}{\partial x_1} \times\left\{\left[x_1^{i_1-1}x_2^{i_2}\ldots x_k^{i_k}\right]Q^{n+1}\left[\ln\left(\frac{Q}{-m^2}\right)-\sum_{k'=1}^{n+1}\frac{1}{k'}\right]\right\} \\ &\quad -\frac{i_1-1}{(n+1)!}\int_0^1 \mathrm{d}x_1\ldots\mathrm{d}x_k\left[x_1^{i_1-2}x_2^{i_2}\ldots x_k^{i_k}\right]Q^{n+1} \times\left[\ln\left(\frac{Q}{-m^2}\right)-\sum_{k'=1}^{n+1}\frac{1}{k'}\right]. \end{aligned} \tag{153}$$

With the identity

$$\begin{aligned} \left(1-\sum_{j=2}^{k}x_j\right)^{i_1-1} &= \sum_{l_2=0}^{i_1-1}\sum_{l_3=0}^{l_2}\ldots\sum_{l_k=0}^{l_{k-1}}\mathcal{C}^{i_1}_{l_2\ldots l_k}x_k^{l_k}x_{k-1}^{l_{k-1}-l_k}\ldots x_2^{l_2-l_3}, \\ \mathcal{C}^{i_1}_{l_2\ldots l_k} &= \frac{(-1)^{l_2}\left(i_1-1\right)!}{\left(i_1-l_2-1\right)!\left(l_2-l_3\right)!\left(l_3-l_4\right)!\ldots l_k!}, \end{aligned} \tag{154}$$

we can write

$$I = \left(1-\delta_{1,k}\right) \times\left\{\sum_{l_2=0}^{i_1-1}\sum_{l_3=0}^{l_2}\ldots\sum_{l_k=0}^{l_{k-1}}\mathcal{C}^{i_1}_{l_2\ldots l_k}\eta''^{(n+1)}_{i_2+l_2-l_3,\ldots,i_k+l_k}\right.$$

$$- \delta_{1,i_1} \eta'^{(n+1)}_{i_2,\ldots,n_k} \Bigg\}$$

$$- (i_1 - 1) \, \eta^{(n+1)}_{i_1-2,i_2,\ldots,i_k}$$

$$- \frac{\delta_{1,k} \left(1 - \delta_{1,i_1}\right)}{(n+1)!} \left(-m^2\right)^{n+1} \sum_{k'=1}^{n+1} \frac{1}{k'}, \qquad (155)$$

where δ represents a Kronecker delta symbol. Finally, we get a recurrence relation

$$p_1^2 \eta^{(n)}_{i_1,\ldots,i_k} + (p_1 \cdot p_2) \, \eta^{(n)}_{i_1-1,i_2+1,i_3,\ldots,i_k} + \ldots$$
$$+ (p_1 \cdot p_k) \, \eta^{(n)}_{i_1-1,i_2,\ldots,i_{k-1},i_k+1}$$
$$= \frac{\delta_{1,k} \left(1 - \delta_{1,i_1}\right)}{2\,(n+1)!} \left(-m^2\right)^{n+1} \sum_{k'=1}^{n+1} \frac{1}{k'}$$
$$- \frac{\left(1 - \delta_{1,k}\right)}{2}$$
$$\times \Bigg\{ \sum_{l_2=0}^{i_1-1} \sum_{l_3=0}^{l_2} \cdots \sum_{l_k=0}^{l_{k-1}} \mathcal{C}^{i_1}_{l_2 \ldots l_k} \eta''^{(n+1)}_{i_2+l_2-l_3,\ldots,i_k+l_k}$$
$$- \delta_{1,i_1} \eta'^{(n+1)}_{i_2,\ldots,n_k} \Bigg\}$$
$$+ \frac{1}{2} \left(i_1 - 1\right) \eta^{(n+1)}_{i_1-2,i_2,\ldots,i_k} + \frac{1}{2} p_1^2 \eta^{(n)}_{i_1-1,i_2,\ldots,i_k}. \qquad (156)$$

The corresponding relations involving the $\xi^{(n)}_{i_1,\ldots,i_k}$ functions are obtained from the equation above by a differentiation, as clearly shown in (149). For completeness we show also the result

$$p_1^2 \xi^{(n)}_{i_1,\ldots,i_k} + (p_1 \cdot p_2) \, \xi^{(n)}_{i_1-1,i_2+1,i_3,\ldots,i_k} + \ldots$$
$$+ (p_1 \cdot p_k) \, \xi^{(n)}_{i_1-1,i_2,\ldots,i_{k-1},i_k+1}$$
$$= \frac{\left(1 - \delta_{1,k}\right)}{2\mathcal{N}}$$
$$\times \Bigg\{ \sum_{l_2=0}^{i_1-1} \sum_{l_3=0}^{l_2} \cdots \sum_{l_k=0}^{l_{k-1}} \mathcal{C}^{i_1}_{l_2 \ldots l_k}$$
$$\times \left[\left(1 - \delta_{0,n}\right) \xi''^{(n-1)}_{i_2+l_2-l_3,\ldots,i_k+l_k} - \delta_{0,n} \eta''^{(0)}_{i_2+l_2-l_3,\ldots,i_k+l_k} \right]$$
$$- \delta_{1,i_1} \left[\left(1 - \delta_{0,n}\right) \xi'^{(n-1)}_{i_2,\ldots,i_k} - \delta_{0,n} \eta'^{(0)}_{i_2,\ldots,i_k} \right] \Bigg\}$$
$$- \frac{(i_1 - 1)}{2\mathcal{N}} \left[\left(1 - \delta_{0,n}\right) \xi^{(n-1)}_{i_1-2,i_2,\ldots,i_k} - \delta_{0,n} \eta^{(0)}_{i_1-2,i_2,\ldots,i_k} \right]$$
$$+ \frac{\delta_{1,k} \left(1 - \delta_{1,i_1}\right) \left(1 - \delta_{0,n}\right)}{2\mathcal{N} \left(-m^2\right)^n} + \frac{1}{2} p_1^2 \xi^{(n)}_{i_1-1,i_2,\ldots,i_k}, \qquad (157)$$

with

$$\mathcal{N} = \begin{cases} n^{-1} \text{ if } n > 0, \\ 1 \text{ if } n = 0. \end{cases}$$

Relations like the left hand side of (156) and (157) naturally emerge when we contract a calculated amplitude with an external momentum. As shown before, recursive use of the above expression makes the verification of the Ward identities an easier task. Even if anomalies are present, by applying the above relations, the expected anomalous term emerges in a natural way [25].

However, note that the above relations do not represent the whole set of possibilities. Similar relations could be found using the momentum interchanging symmetry of the $\eta^{(n)}_{i_1,\ldots,i_k}$ (and $\xi^{(n)}_{i_1,\ldots,i_k}$) functions

$$\eta^{(n)}_{i_1,\ldots,i_k} \left(p_1, \ldots, p_k\right) = \eta^{(n)}_{i_j,\ldots,i_{j-1},i_1,i_{j+1},\ldots} \left(p_1 \longleftrightarrow p_j\right),$$

as we have seen in detail in Sect. 4. If we perform this operation in (157) we get a system of equations which we can solve showing that all n-point functions can be reduced, in the end, only to functions with $i_1 = i_2 = \ldots = i_k = 0$. Explicitly we have to solve the following system of linear equations:

$$\begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & \ldots & p_1 \cdot p_k \\ p_2 \cdot p_1 & p_2 \cdot p_2 & \ldots & p_2 \cdot p_k \\ \vdots & \vdots & \ddots & \vdots \\ p_k \cdot p_1 & p_k \cdot p_2 & \ldots & p_k \cdot p_k \end{pmatrix}$$
$$\times \begin{pmatrix} \eta^{(n)}_{i_1,\ldots,i_k} \\ \eta^{(n)}_{i_1-1,i_2+1,i_3,\ldots,i_k} \\ \vdots \\ \eta^{(n)}_{i_1-1,i_2,\ldots,i_{k-1},i_k+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix},$$

or, in a compact notation, $\mathbf{A}\xi = \mathbf{B}$ where $a_{ij} = p_i \cdot p_j$ and

$$b_1 \left(p_1, \ldots, p_k\right)$$
$$= \frac{\delta_{1,k} \left(1 - \delta_{1,i_1}\right)}{2\,(n+1)!} \left(-m^2\right)^{n+1} \sum_{k'=1}^{n+1} \frac{1}{k'}$$
$$- \frac{\left(1 - \delta_{1,k}\right)}{2}$$
$$\times \Bigg\{ \sum_{l_2=0}^{i_1-1} \sum_{l_3=0}^{l_2} \cdots \sum_{l_k=0}^{l_{k-1}} \mathcal{C}^{i_1}_{l_2 \ldots l_k} \eta''^{(n+1)}_{i_2+l_2-l_3,\ldots,i_k+l_k}$$
$$- \delta_{1,i_1} \eta'^{(n+1)}_{i_2,\ldots,n_k} \Bigg\}$$
$$+ \frac{1}{2} \left(i_1 - 1\right) \eta^{(n+1)}_{i_1-2,i_2,\ldots,i_k} + \frac{1}{2} p_1^2 \eta^{(n)}_{i_1-1,i_2,\ldots,i_k}, \qquad (158)$$

or, in general, $b_j \left(p_1, \ldots, p_k\right) = b_1(p_j \leftrightarrow p_1)$ with $i_1 = i_j + 1$ and $i_j = i_1 - 1$. Inverting the matrix $\mathbf{A}$ we get the following recurrence relation:

$$\eta^{(n)}_{i_1,\ldots,i_k} = \sum_{j=1}^{k} a^{-1}_{1j} b_j = \sum_{j=1}^{k} \frac{\Delta_{1j} b_j}{\det \mathbf{A}}, \qquad (159)$$

where Δ_{ij} is the cofactor of a_{ij} which can be easily obtained for each specific number of points in a kinematical situation with $\det \mathbf{A} \neq 0$. Specifically, setting $n = 0, k = 3, i_1 = 1, i_2 = i_3 = 0$ in (159) we get the reduction (29) and so on. By recursive use of the above relation it is possible to reduce all functions $\eta^{(n)}_{i_1,\ldots,i_k}$ to functions with $i_1 = i_2 = \ldots = i_k = 0$. This type of reduction is useful, for example, in applications where we are interested in numerical results because within this procedure we have to manipulate only a low number of mathematical structures saving, in this way, considerable computational time.

10 Conclusions

In the present contribution we considered a systematization for the evaluation of Feynman integrals which are typical of perturbative calculations in QFT's. In the proposed strategy, it is possible to avoid the use of an explicit regulator in intermediary steps and the calculations are performed by taking arbitrary choices for the momenta of the internal lines. With this attitude two important types of arbitrariness are preserved in the calculations; the choice of the regularization and the choice of internal lines momenta labels. Given this fact the results for the considered integrals can be converted to the corresponding ones of any specific regularization as well as other philosophies like that which consider the evaluation of surface terms in divergent physical amplitudes. In the proposed calculational strategy two types of systematization are introduced: one for the divergent parts, and another one for the finite ones.

The divergent content of all Feynman integrals are written in terms of five objects. Two of them are the basic divergent objects $I_{\log}(m^2)$ and $I_{\text{quad}}(m^2)$ which are irreducible and will invariably be absorbed in the renormalization, independent of the mathematical form assumed which will depend on the specific regularization adopted or could be fitted by physical observables in effective nonrenormalizable theories [24]. The remaining three divergent standard objects $\square_{\alpha\beta\mu\nu}$, $\Delta_{\mu\nu}$ and $\nabla_{\mu\nu}$ are differences between divergent integrals of the same degree of divergence. Their values are dependent on the specific regularization or the equivalent philosophy adopted. The relevant dependence of a perturbative calculation on the regularization resides in the value attributed to these three objects. Concerning the ambiguities associated to the arbitrariness involved in the choice of labels for the internal lines momenta, the situation is also very transparent. Only terms involving the objects $\square_{\alpha\beta\mu\nu}, \Delta_{\mu\nu}$ and $\nabla_{\mu\nu}$ are potentially ambiguous. The analyses of Ward identities, on the other hand, revealed that all potentially violating terms are proportional to the objects $\square_{\alpha\beta\mu\nu}, \Delta_{\mu\nu}$ and $\nabla_{\mu\nu}$ even if they are not always ambiguous.

The systematization for the finite parts is made by the introduction of a set of structure functions, one for each number of propagators. For two-point functions all the results can be written in terms of $Z_k(p^2; m^2)$ which can be reduced to $Z_0(p^2; m^2)$. For integrals having three propagators the systematization is obtained through the structure functions $\xi_{nm}(p, q)$ and $\eta_{nm}(p, q; m^2)$. These two sets of functions are also connected. In fact, all the functions $\eta_{nm}(p, q; m^2)$ can be written in terms of $\xi_{nm}(p, q)$ which, in the end, can be reduced to only $\xi_{00}(p, q)$. This means that all three-point functions can be written in terms of $\xi_{00}(p, q)$ and $Z_0(p^2; m^2)$. These reductions, on the other hand, involve relations among the functions $\xi_{nm}(p, q)$, $\eta_{nm}(p, q; m^2)$ and $Z_k(p^2; m^2)$ and combinations of such relations are crucial for the verification of relations among Green's function or Ward identities having three and two propagators. In the case of Feynman integrals having four propagators, the systematization is obtained through the structure functions which we call $\zeta_{nml}(p, q, r)$, $\xi_{nml}(p, q, r)$ and $\eta_{nml}(p, q, r; m^2)$. The last two functions can be related to $\zeta_{nml}(p, q, r)$ which can be reduced to only $\zeta_{000}(p, q, r)$. This means that all four-point functions of the perturbative calculations can be written in terms of $\zeta_{000}(p, q, r)$, $\xi_{00}(p, q)$ and $Z_0(p^2, m^2)$. Combinations of such relations are crucial properties when the verification of relations among Green's functions and Ward identities involving four and three points are considered. These systematizations can be easily extended to higher number of points, as shown explicitly in Sect. 9 where a set of recurrence formulae was presented which unify the notation and generalize the relations and reductions of finite structure functions considered in Sect. 4.

The convenience of the above mentioned systematization has been shown in Sect. 6, where the evaluation of physical amplitudes of the perturbative calculations was considered and, in Sects. 7 and 8, where the relations among the involved Green's functions are verified. It was shown that the calculations can be performed by taking the most general expression for the involved amplitudes. A clear and transparent description of the potentially ambiguous and symmetry violating terms as well as their regularization dependence character is obtained. Besides, such decompositions emphasizes in a very clear way physical aspects relative to unitarity which are contained in the perturbative amplitudes and, in addition, allows a simple systematization for the study of kinematical limits of physical interest. Another aspect, which is important to emphasize, is the dimensional extension of the adopted procedure. In every previously chosen dimensions it is possible to identify the basic divergent objects and the structure functions for the finite parts of the n-point Green's functions, and to construct the analogous reductions and relations identified in the present discussion to consider the relations among Green's functions and Ward identities associated with them [23]. In the same way, the extension to more than one loop is perfectly possible. New basic divergent structures and other classes of structure function will appear.

Finally, we would like to call attention to the universal character of the proposed treatment of divergences in perturbative calculations of QFT's. In fact, a divergent integral is not really evaluated. Only general properties for the divergent integrals are required to be assumed in order to evaluate physical amplitudes in a consistent way (ambiguity free and symmetry preserving). No explicit form of regularization needs to be assumed for any purposes in perturbative calculations. Every amplitude can be viewed on the same footing and treated in an unique way. The success of regularization prescriptions, as well as their inconsistencies, can be clearly understood within the proposed strategy. Generalizations for more general cases, like that involving different masses for the propagators and two loops as well as for other space-time dimensions, are presently under way. In the investigations presently performed, all the results are in perfect agreement with our expectations.

Acknowledgements. The authors acknowledge CNPq/Brazil.

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